Pontifícia Universidade Católica
do Rio de Janeiro

Rafael Pereira Alves

Does the Stock Market reflect the Long-Run Effects of COVID-19?

Dissertação de Mestrado

Thesis presented to the Programa de Pós–graduação em Economia, do Departamento de Economia da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Economia.

Advisor: Prof. Walter Novaes Filho

Rio de Janeiro
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Abstract

Alves, Rafael Pereira; Novaes Filho, Walter (Advisor). **Does the Stock Market reflect the Long-Run Effects of COVID-19?**. Rio de Janeiro, 2022. 63p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

The existing literature on the effects of Covid on stock returns focuses on endogenous changes in risk tolerance and on the modeling of rare events. So far, these attempts have not been able to match the data. In this paper, I propose an alternative approach to explaining the Covid effects on stock returns worldwide: disentangling the long-run effects from the short-run effects. Intuitively, Covid's long-run effects include disruptions of supply chains and educational patterns, which, conceivably, will take time to phase out. Exactly as it happens with the persistent shocks of long-run risks models! A model that allows for short-run fluctuations and long-run risk shows that persistent shocks play a role in explaining stock market returns and exchange rates in a time span that starts in January 2018 and ends in November 2021.

Keywords

Long-Run Risks; COVID-19; Stock Market Returns; Exchange Rates.

Resumo

Alves, Rafael Pereira; Novaes Filho, Walter. **O Mercado Acionário Reflete os Efeitos de Longo Prazo da COVID-19?**. Rio de Janeiro, 2022. 63p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

A literatura existente sobre os efeitos da Covid nos retornos das ações concentra-se em mudanças endógenas na tolerância ao risco e na modelagem de eventos raros. Até agora, essas tentativas não foram capazes de corresponder aos dados. Neste artigo, proponho uma abordagem alternativa para explicar os efeitos da Covid nos retornos de ativos em todo o mundo: separar os efeitos de longo prazo dos efeitos de curto prazo. Intuitivamente, os efeitos de longo prazo da Covid incluem disrupções nas cadeias produtivas e padrões educacionais, que, concebivelmente, levarão tempo para serem eliminados. Exatamente como acontece com os choques persistentes dos modelos de risco de longo prazo! Um modelo que permite flutuações de curto prazo e risco de longo prazo mostra que choques persistentes desempenham um papel na explicação dos retornos do mercado de ações e das taxas de câmbio em um período de tempo que começa em Janeiro de 2018 e termina em Novembro de 2021.

Palavras-chave

Riscos de Longo Prazo; COVID-19; Retornos do Mercado Acionário; Taxa de Câmbio.

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There is no terror in the bang, only in the anticipation of it.

Alfred Hitchcock, *Halliwell's Filmgoer's Companion.*

1
Introduction

Covid has had an impressive impact on stock markets. Between March 2020 and April 2020, the MSCI World Index lost 34% of its value, while FTSE All-Share Index fell by 33%.[1] As seen in Figure 1.1, the stock market index of Brazil, Ibovespa, lost around 50% of its value. Besides being the most serious health crisis since the Spanish Flu in 1918, nearly a century ago, Covid is the most imposing shock to hit markets since the Great Financial Crisis of 2008. Given that, it's entirely expected that this event would attract the attention of academics, seeking to understand the impact of Covid – and the health policies implemented during its combat – on economic activity. In particular, to undertake the difficult task of explaining stock returns during the pandemic.

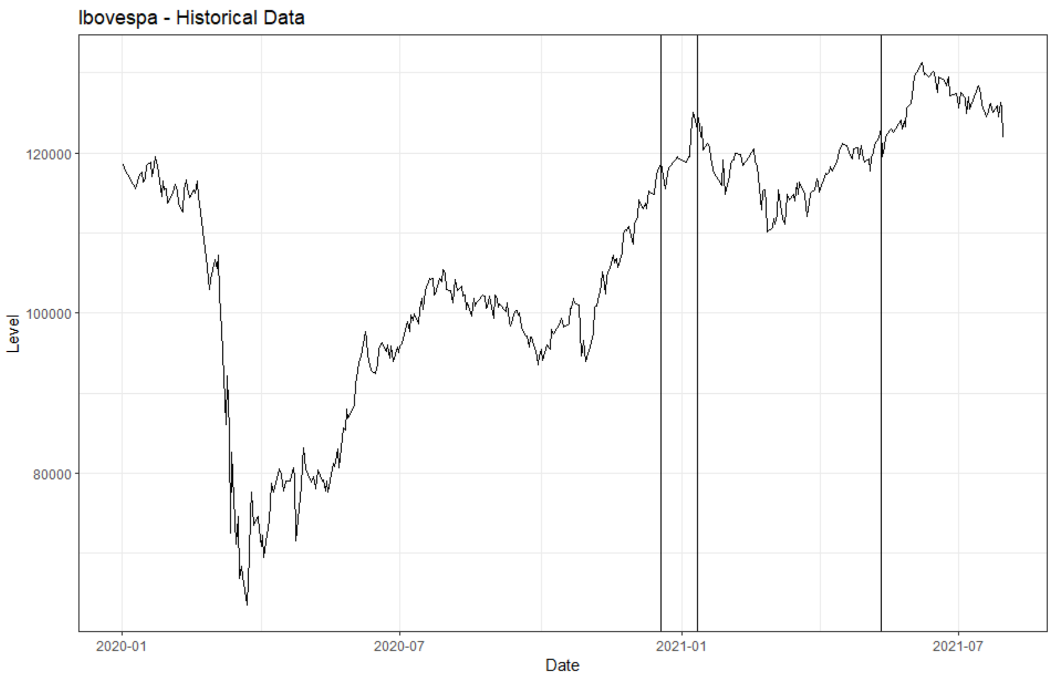
The literature has tried to explain the 2020 stock market crash and the market after Covid by relying essentially on two mechanisms: The modelling of rare events and endogenous changes in risk tolerance. The first approach models the consumption process being subject to rare, intense short-run negative shocks, which occur given a definite probability law. Examples of this type of mechanism include Barro (2006), Gabaix (2012), Martin (2013) and Wachter (2013). The second relies on sentiments shocks, which affect the risk preference of economic agents, such as Chau, Deesomsak, and Koutmos (2016).

As per Davis, Lui and Chang (2021) and Gormsen and Koijen (2020), the rare disaster model cannot explain the stock market reaction to Covid: The short-run impact on economic activity, while very impressive, would have to be an order of magnitude larger to explain stock returns. On the other hand, Cox, Greenwald and Ludvigson (2020) and Caballero and Simsek (2021) try the second approach, with moderate success.

There is, however, a problem in relying on an endogenous risk-tolerance mechanism to explain the Covid stock market: The empirical evidence points to no changes in risk aversion during the pandemic. Examples in this literature include Angrisani et al. (2020) and Drichoutis and Naiga (2021). Given that,

[1]Davis, Jonathan (2020). The Investment Trusts Handbook 2021. Harriman House. p. 96. ISBN 9780857198952.

Figure 1.1: Ibovespa Index - Historical Data and Announcement Dates of Variants of Concern

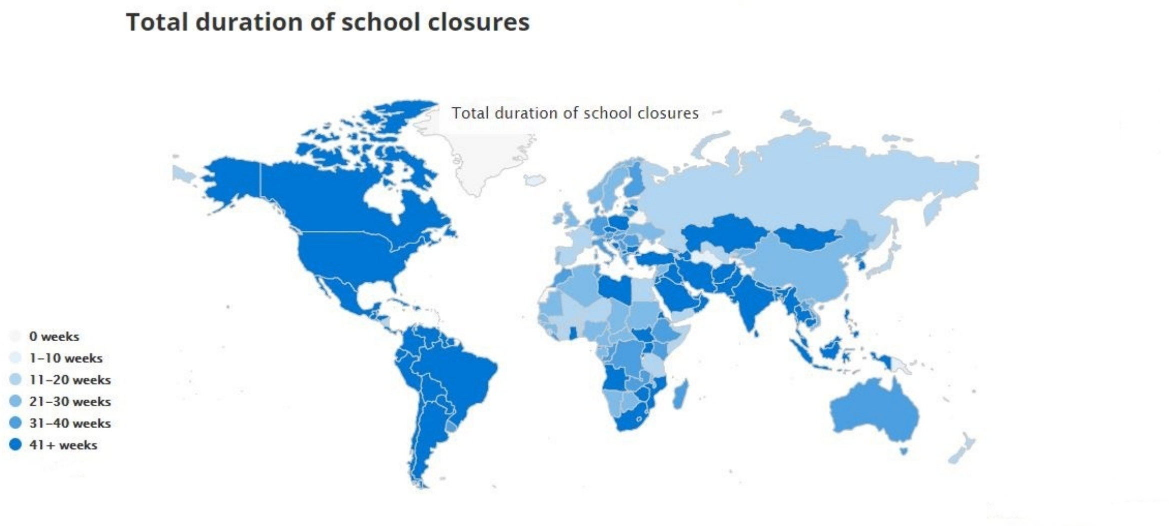


it is unreasonable to assume such a mechanism as a possible explanation of this event.

I propose to tackle this problem by taking into consideration the long-run effects of Covid and the economic uncertainty promoted by this event. The long-run risks model of Bansal and Yaron (2004) has this ingredients, since it incorporates long-run effects on consumption and stochastic volatility as its key features. Moreover, the LRR model has been successfully applied to a plethora of different problems in the asset pricing literature. Examples include Piazzesi and Schneider (2007), which investigate the term structure of the interest rate; Chen (2010), applying it to the credit spread; Dreschler and Yaron (2011) and Eraker and Shaliastovich (2008), who develop implications for option pricing. A summary of this literature and other topics on Macro-Finance can be found in Cochrane (2017).

So what do I mean by the long-run effects of Covid? There are two economically motivated intuitions behind this. First, one may think about the supply chain disruptions during Covid. As a quick anecdote, one has to wait a few months in line in order to purchase a car in Brazil. Second, there's the impacts on human capital acquisition that are due to lockdown policies, which were necessary to contain the spread of the virus.

Figure 1.2: Total Duration of School Closures



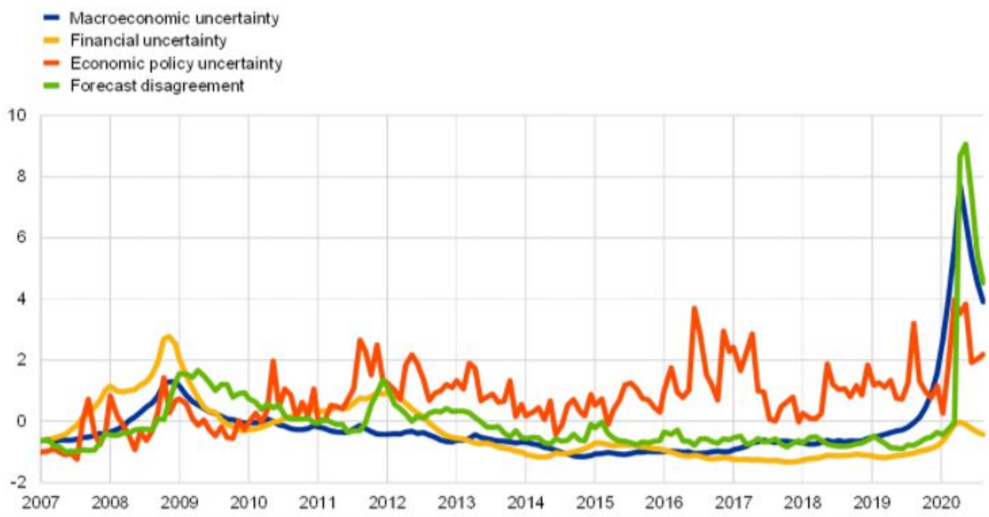
In Figure 1.2 one may see the total duration of school closures around the world. Two things are immediately noticeable: First, that there's heterogeneity in educational outcomes during Covid. Some countries, like Brazil, had closures that took over 41 weeks to end, while others adopted this policy for a much shorter duration. Second, the impact of school closures is very significant. Given the prior educational structure of each country, human capital accumulation was heavily affected by social distancing policies. Countries in Latin America, for instance, have poor basic education infrastructure, and were much less capable to adapt to online learning. Given that human capital is essential to future economic productivity, this should signify a persistent shock on consumption.

Apart from the aforementioned facts, there is also a general increase in economic uncertainty, which has implications both for economic activity and for the stock market. Figure 1.3 shows the evolution, as a deviation from the historical average, of uncertainty measures in the Euro Zone. We observe a very significant increase in uncertainty as a result of the Covid crisis. This uncertainty surpasses, for almost all measures, the impact of the subprime crisis.

Another interesting feature of Covid is that it mutates. [2] Outside of the initial March 2020 shock, the stock market has reacted throughout later 2020 and 2021 to its variants. The vertical lines in Figure 1.1 represent the

[2]https://www.bbc.com/future/article/20210127-covid-19-variants-how-mutations-are-changing-the-pandemic

Figure 1.3: Standard deviation of uncertainty measures in the Eurozone. (Source: ECB Economic Bulletin, Issue 6/2020)



announcements of variants of concern by the World Health Organization (WHO). One can readily see that variants have been a source of volatility in the stock market, producing adverse reactions that, while not as intense as the crash of March 2020, are still quite significant.

There are several difficulties in explaining stock markets during Covid: First, being a rare event, it imposes a series of statistical problems, such as breaking of ergodicity. Second, the stock market reaction may be partially explained by changes in risk tolerance, or myopic investors. Third, the very act of modelling a stochastic discount factor (SDF) presents a series of difficulties – it is not obvious what mechanisms could be behind the stock market reaction to Covid. Finally, the pandemic period in itself is relatively short – only two years of data.

In order to deal with the problems of studying stock market returns during Covid, I modify the standard long-run risks model to disentangle short-run business cycle shocks and long-run persistent shocks. The model is able to capture persistent – yet small – long-run risks and larger, less persistent short-run business cycle shocks. Moreover, I work with and open economy version of the model, inspired by Colacito and Croce (2011). This helps, since the impact of both short and long-run shocks related to Covid are heterogeneous across countries. This heterogeneity helps to pin down the long-run effects of Covid implied in data.

I do a particle Bayesian MCMC estimation of the dynamics of the model, using data from two countries: Brazil and the United States. For both countries, I gathered data on market returns, the risk-free rate, inflation rate (to deflate variables), consumption and dividends yields. The sample consists of monthly observations, spanning January 2018 to November 2021 for both countries, giving a total of 47 observations for each variable in a given country.

To solve for asset prices, I rely on two methods: (1) A log-linearized solution around the steady state, which has been the staple method in the literature since Bansal and Yaron (2004). (2) A local projection method, first proposed in Pohl, Schmedders and Wilms (2018). I show that, corroborating the latter, the local projection method gives a closer fit to returns and exchange rate data.

This model is successful in matching moments of stock market returns, risk-free returns and exchange rate depreciation in this time period. Given the persistence of long-run shocks, I show that long-run risks are an important element driving asset prices. For instance, the half-life of a long-run shock on consumption increases from 18 months before to 42 months after the pandemic in Brazil. Moreover, I show that the state variable associated with long-run consumption risk becomes more correlated between countries during Covid (from 0.368 to 0.633), which further corroborates the importance of long-run risks associated with Covid in determining asset prices during the pandemic. Sadly, the model is unable to track asset return dynamics. However, I show that including long-run risks improves model performance in this dimension.

Therefore, in this paper I present a long-run risks explanation for the stock market reaction during Covid. To the best of my knowledge, this is the first study to employ this model to this particular problem.

This thesis goes on as follows: Section 2 I describe the model framework, highlighting agents preferences, the underlying dynamics of the economy and how to derive the exchange rate. Section 3 describes the dataset that was collected in order to undertake this study. It describes the econometric framework used to estimate the posterior median of each parameter, and also the calibration for preference parameters. Section 4 has a presentation of the results, including posterior medians and standard deviation for each parameter, and also asset pricing implications of the model under a log-linearized solution and local projection methods. Section 5 concludes the paper, summarizing its aims, findings and possibilities for further work.

2
Model

I consider an economy with two countries: Brazil and the United States. It is not immediate how one should model their interaction. Countries may be self-sufficient, which would imply no difference in estimating countries jointly or separately. They may have perfectly integrated economies, which would mean that they share the same vector of shocks. Finally, one may consider that countries are not perfectly integrated, but neither are self-sufficient. In this case, shocks are cross-country correlated. I choose this last scenario as my benchmark. Following Colacito and Croce (2011), I name this benchmark scenario as countries operating under autarky. [1]In Section 4.2 I argue that this base scenario is the one who best fits data.

2.1. Preferences and Stochastic Discount Factor

Agents from both countries have Epstein-Zin-Weil preferences, as in Epstein and Zin (1989) and Weil (1989). The agent of each country has a utility function recursively given by:

$$U_t(C) = \left((1-\beta)C_t^{1-\theta} + \beta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{1}{1-\theta}}, \tag{2-1}$$

where C_t is consumption at time t, β is the agents time preference parameter, γ is his relative risk aversion and $\frac{1}{\theta}$ is the elasticity of intertemporal substitution (EIS). These preferences allow for a separation between risk aversion and the EIS, and have helped reconcile consumption based asset pricing models with stock market data (see Bansal and Yaron (2004)).

The agent maximize his utility function subject to the following budget constraint:

$$W_{t+1} = (W_t - C_t)\,R_{w,t+1}, \tag{2-2}$$

where W_t is the agents wealth and $R_{w,t+1}$ the return on the wealth portfolio.

[1]Admittedly, one may find this definition perplexing, and consider that the fist scenario is the true autarky. I chose to follow this nomenclature – which means that one is not modelling the endowment process in each economy as containing a common component – just to be consistent with the literature.

For any asset a with ex-dividend price $P_{a,t}$ and dividend $D_{a,t}$, the standard Euler equation holds,

$$\mathbb{E}_t\left[M_{t+1}R_{a,t+1}\right] = 1. \tag{2-3}$$

Where $R_{a,t+1} = \left(P_{a,t+1} + D_{a,t+1}\right)/P_{a,t}$ is the return of asset a and M_{t+1} is the stochastic discount factor (SDF). The risk-free rate in this economy is given by $R_{f,t} = 1/\mathbb{E}_t\left[M_{t+1}\right]$. Moreover, the SDF can be written as:

$$M_{t+1} = \beta^{(1-\gamma)/(1-\theta)}\left(\frac{C_{t+1}}{C_t}\right)^{-\theta(1-\gamma)/(1-\theta)} R_{w,t+1}^{(\theta-\gamma)/(1-\theta)}. \tag{2-4}$$

Epstein and Zin (1991) show that the logarithm of the SDF, m_{t+1} can be written as:

$$m_{t+1} = \frac{1-\gamma}{1-\theta}\log(\beta) - \frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c_{t+1} + \frac{\theta-\gamma}{1-\theta} r_{c,t+1}, \tag{2-5}$$

where Δc_{t+1} is the log consumption growth and $r_{c,t+1}$ the log return on the asset that pays consumption.[2]

In the following section, I specify the exogenous laws of motion for consumption and dividend growth rates in each economy, which completes the system.

2.2. Consumption and dividend process

The growth rate of consumption in a given country is given by:

$$\Delta c_{t+1} = \mu_c + x_t + \Delta z_{t+1} + \sigma_t \epsilon_{c,t+1}, \tag{2-6}$$

where μ_c is the mean growth of consumption; x_t is the long-run risks process, an AR(1) process with persistence ρ_x between 0 and 1. z_t is the business cycle component, with persistence ρ_z, also in the unit interval. Their laws of motion are given by

$$\begin{aligned} x_t &= \rho_x x_{t-1} + \varphi_x \sigma_t \epsilon_{x,t} \\ \Delta z_{t+1} &= \rho_z \Delta z_t + \varphi_z(\epsilon_{z,t+1} - \epsilon_{z,t}). \end{aligned} \tag{2-7}$$

The first process captures small and highly persistent shocks on consumption, while the other captures short run deviations form potential consumption, less persistent but frequently larger in magnitude.

[2]This is equivalent to the log of the return on the wealth portfolio, that is, $r_{c,t+1} = r_{w,t+1}$.

These parameters are crucial for asset pricing dynamics in the model: As Bansal and Yaron (2004) point out, ρ_x tells us how truly relevant are long-run risks. We should expect a value close to one. ρ_z on the other hand, should be smaller, given that it capture short-run, less persistent shocks. In Section 4.1 I show that this is indeed the case, and that long-run risks become even more important for asset pricing dynamics during the pandemic.

State variables determine the dividend process by λ_x and λ_z. λ_x maps the long-run state variable x, while λ_z maps the short-run state variable z. μ_d is the mean growth of dividends:

$$\Delta d_{t+1} = \mu_d + \lambda_x x_t + \lambda_z \Delta z_{t+1} + \varphi_d \sigma_t \epsilon_{d,t+1}. \tag{2-8}$$

The consumption growth, dividends and long-run risks are influenced by stochastic volatility, σ_t^2. This process has mean σ^2, persistence ρ_s and captures the time-varying economic uncertainty in this economy:

$$\sigma_t^2 = \sigma^2 + \rho_s \left(\sigma_{t-1}^2 - \sigma^2\right) + \varphi_s \epsilon_{s,t}. \tag{2-9}$$

Finally, I summarize the vector of shocks in this economy. I use the asterisk to refer to variables in a foreign country. I allow shocks to be cross-country correlated,

$$\xi_t = \left[\epsilon_{c,t}, \epsilon_{d,t}, \epsilon_{x,t}, \epsilon_{z,t}, \epsilon_{s,t}, \epsilon_{c,t}^*, \epsilon_{d,t}^*, \epsilon_{x,t}^*, \epsilon_{z,t}^*, \epsilon_{s,t}^*\right]. \tag{2-10}$$

The shocks on this economy and the full system of equations, 2-5 to 2-10, will be estimated using data from Brazil and the US, applying particle Bayesian MCMC methods akin to Fulop et al (2022), which are introduced in Section 3.2 and further described in Appendix B. Bayesian methods are useful in this setting, since they are robust to small sample size, which is an unavoidable problem of empirical studies of Covid.

Asset prices are solved for using two different methods: (1) A log-linearized solution around the steady-state, as proposed in Bansal and Yaron (2004) and used extensively in the literature (for instance, see Bansal, Kiku and Yaron (2012); Beeler and Campbell (2012); Schorfheide, Song, and Yaron (2018)).[3] (2) A local projection method, as proposed in Pohl, Schmedders and Wilms (2018), which have shown that log-linearization may lead to large, economically

[3]The numerical solution for the log-linearized model consists simply of finding a fixed-point solution for the price-consumption and price-dividend ratio. Given those two variables and the model parameters, one may find the remaining variables and asset pricing moments.

significant errors for model-implied returns, given the intrinsically nonlinear nature of long-run risks models. The local projection method is described in Section 3.2.2 and Appendix C.

2.3. Exchange Rates

Given that our estimation methodology relies on using data from Brazil and the United States, we must derive exchange rate dynamics and use them to evaluate the estimated model. The derivation of the exchange rate depreciation in this section follows closely Backus, Foresi and Telmer (2001).

Let E_t be the real (R$) spot price of one dollar. Let R_t^* be the dollar return of one asset. It is true that:

$$\mathbb{E}_t \left[M_{t+1}^* R_{t+1}^* \right] = 1, \tag{2-11}$$

$$\mathbb{E}_t \left[M_{t+1} R_{t+1} \right] = 1, \tag{2-12}$$

where M_t^* is the pricing kernel of the US. We may price the same asset using the pricing kernel of Brazil, M_t. The return in reais for this asset is given by $R_{t+1} = \left(E_{t+1}/E_t \right) R_{t+1}^*$ and,

$$1 = \mathbb{E}_t \left[M_{t+1} \left(E_{t+1}/E_t \right) R_{t+1}^* \right]. \tag{2-13}$$

Combining (15) and (17), we get that:

$$\mathbb{E}_t \left[M_{t+1}^* R_{t+1}^* \right] = \mathbb{E}_t \left[M_{t+1} \left(E_{t+1}/E_t \right) R_{t+1}^* \right]. \tag{2-14}$$

Notice that pricing kernel $M_{t+1}^* = M_{t+1} E_{t+1}/E_t$ trivially satisfies the above equality.

Given what was stated above, one may write $\Delta e_t = m_t^* - m_t$. Therefore, the model implied depreciation rate of the real is simply the log difference of the pricing kernels on the US and Brazil.

In section 4.1 I show that the estimated stochastic discount factor does a good in pricing assets on both economies.

2.4. Log-Linearized Solution

I sketch below the approximate log-linearized solution to the model. Its full exposition can be found in Appendix A. Notice that this log-linearization process differs a bit from standard solving methods used in the Macroeconomics literature.[4] Here I rely on the Campbell-Shiller (1988) approximation for the price-consumption and price-dividend ratios of the economy. Moreover, I use a Guess and Verify method to conjecture a linear relationship between those quantities and sources of risk in the model. Combining both elements, I can arrive at approximate closed form solutions for the asset returns and variances in this economy.

First, I start by finding the solution for the return on the consumption claim, $r_{c,t+1}$. The Campbell-Shiller approximation for this return is given by $r_{c,t+1} = \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}$.[5] I conjecture that the log price-consumption ratio follows $pc_t = A_0 + A_1 x_t + A_2 \Delta z_{t+1} + A_3 \sigma_t^2$. We may substitute both on the Euler condition, recalling that $\Delta c_{t+1} = \mu_c + x_t + \Delta z_{t+1} + \sigma_t \epsilon_{c,t+1}$,

$$E_t \left[\exp \left(\frac{1-\gamma}{1-\theta} \log(\beta) - \frac{1-\gamma}{\frac{1}{\theta} - 1} \Delta c_{t+1} + \frac{\theta - \gamma}{1-\theta} r_{c,t+1} + r_{a,t+1} \right) \right] = 1. \tag{2-15}$$

Exploring the fact that state variables are conditionally log-normal, one may find expressions for constants depending entirely on the model parameters.

With the approximate closed form solution for $r_{c,t+1}$, we may find the SDF depending entirely on state variables and model parameters:

$$\begin{aligned} m_{t+1} - E_t(m_{t+1}) = & \zeta_c \sigma_t \epsilon_{c,t+1} - \zeta_x \sigma_t e_{x,t+1} - \zeta_z (e_{z,t+2} - e_{z,t+1}) \\ & - \zeta_s \varphi_s \epsilon_{s,t+1}. \end{aligned} \tag{2-16}$$

Notice that the pricing kernel is simply a function of each source of risk in this model, where ζ_i is the market price of risk for each source.

The risk premium for any asset is determined by the conditional covariance between the return and m_{t+1}. Similar to $r_{c,t+1}$, I use the Campbell-Shiller

[4]For instance, log-linearization as used in Smets and Wouters (2007).

[5]This result is found by doing a first-order Taylor expansion around the mean price-consumption ratio on the log-return of the consumption claim.

approximation for the price-dividend ratio to find the market return r_t and its variance. The expression for the Equity Premium is given by:

$$E_t\left(r_{t+1} - r_{f,t}\right) = \beta_{pd,x}\zeta_{pd,x}\sigma_t^2 + \beta_{pd,z}\zeta_{pd,z} + \beta_{pd,s}\zeta_{pd,s}\varphi_s^2 - 0.5\,\text{var}_t\left(r_{t+1}\right). \quad (2\text{-}17)$$

With $\beta_{pd,i}$ depending on the model parameters. We may interpret those as the risk loading relating to each source of risk. Notice that it also depends on the unconditional variance of r_t. Thankfully, one may find the close-form solution for this quantity:[6]

$$\begin{aligned}\text{var}\left(r_t\right) =& \theta^2[\text{var}(x_t) + \text{var}(\Delta z_{t+1})] + \left[\beta_{pd,x}^2 + \varphi_d^2\right]\sigma^2 \\ &+ \beta_{pd,z}^2 + \left[A_{3,pd}\left(\rho s \kappa_1 - 1\right)\right]^2 \text{var}\left(\sigma_t^2\right) + \beta_{pd,s}^2\varphi_s^2.\end{aligned} \quad (2\text{-}18)$$

Finally, the risk-free rate can be found using the Euler condition, the variance of m_{t+1} and the risk premium of $r_{c,t+1}$. Its expression is given by:

$$\begin{aligned}E\left(r_{f,t}\right) =& -\log(\beta) + \theta E(\Delta c) + \frac{(\gamma - \theta)}{1-\gamma}E\left[r_{c,t+1} - r_{f,t}\right] \\ &- \frac{1-\theta}{2(1-\gamma)}\left[\left(\zeta_c^2 + \zeta_x^2\right)E\left[\sigma_t^2\right] + \zeta_z^2 + \zeta_s^2\varphi_s^2\right].\end{aligned} \quad (2\text{-}19)$$

The unconditional variance of $r_{f,t}$ is

$$\text{var}\left(r_{f,t}\right) = \theta^2[\text{var}\left(x_t\right) + \text{var}\left(\Delta z_{t+1}\right)] + \left\{\frac{(\gamma-\theta)}{1-\gamma}Q_1 - Q_2\frac{1-\theta}{2(1-\gamma)}\right\}^2 \text{var}\left(\sigma_t^2\right). \quad (2\text{-}20)$$

Again, Q_1 and Q_2 are constants, which depend on the model parameters. As noted in Bansal and Yaron (2004), for all practical purposes, the variance of the risk-free rate is determined by the first term.

[6] $A_{3,pd}$ is a constant, which depends on the model parameters. Its value can be found in Appendix A.

3
Data and Econometric Framework

3.1. Data Sources and Calibrated Parameters

I use data from two countries: Brazil and the United States. For both countries, I gathered data on market returns, the risk-free rate, inflation rate (to deflate variables), consumption and dividends yields. The sample consists of monthly observations, spanning January 2018 to 2021 for both countries.

For Brazil, I gathered monthly market returns from IBrX 100,[1] collected from the Reuters Datastream platform. Monthly risk-free (Selic) rates were found at the BCB (Brazil's Central Bank) database. From IBGE I collected the monthly IPCA, the consumer's price index. I collected the monthly consumption data from Fecomercio SP. This variable proxies monthly consumption data in Brazil using a measure from São Paulo. Sadly, there is no direct measure of monthly consumption for Brazil, so this compromise was necessary. Finally, dividend yield data comes from NEFIN, the Center for Research in Financial Economics of the University of São Paulo.

The United States data also comes from various sources: Monthly market (S&P 500) returns were collected at the Reuters Datastream platform; the risk-free rate was collected from Kenneth French's database. From the US Bureau of Labor Statistics I extracted monthly CPI rates. Monthly consumption data can be found at the FRED St.Louis Database. Dividend yield was gathered at Robert Shiller's online database.

In order to estimate the model, I calibrate a few parameters, relating to the agent's preferences. I chose standard values in the literature, which can be found in Table 1. One may be curious if the subjective time discount factor can credibly be that large for Brazil. Given that the model consists of a monthly decision problem, the parameter is very close to 1. For the year of 2019,[2] the average 1-month realized real interest rate was 0.17%.[3] Given that, one may

[1]The results of this paper are robust to substituting the IBrX 100 with the Ibovespa Index.

[2]I choose the year 2019 as the GDP per capita growth was approximately 0 that year.

[3]The 1-year nominal yearly interest rate was 6,51%, while the yearly inflation rate was 4,02%.

roughly estimate $\hat{\beta} = \frac{1}{1+i^{t\to t+1}} = \frac{1}{1.0017} \approx 0.998$.

Table 3.1: Calibrated Parameters

Parameter	Description	Value
β	Subjective Discount Factor	0.998
γ	Coefficient of Relative Risk Aversion	6
$\frac{1}{\theta}$	Intertemporal Elasticity of Substitution	2

For the Intertemporal Elasticity of Substitution (IES), I chose a value of 2. This value, being greater than 1, ensures that stock prices rise with expected future consumption growth and fall with volatility of consumption growth. There is a lot of disagreement with respect to the IES true value: estimates range from negative values and up to 10, as shown in Havranek et al (2015). While this calibration is slightly larger than the value of 1.5 used in Bansal and Yaron (2004), it is well within the range of empirical estimates found in the literature. Moreover, this calibration is very similar to the posterior median of 1.97 found in Schorfheide, Song and Yaron (2018), who also utilize Bayesian methods to estimate a Long-Run Risks model.

3.2. Econometric Framework

It is instructive to think of the methodology applied to this study as containing two main blocks. First, Bayesian methods are employed in order to estimate the parameters that govern the dynamics of the economy. As described in Section 3.1, the subjective discount factor β, the coefficient of risk aversion γ and the intertemporal elasticity of substitution $\frac{1}{\theta}$ are calibrated. Estimated parameters can be summarized (for each country) by $\Theta = [\rho_z, \rho_x, \rho_s, \mu_c, \mu_d, \sigma, \varphi_z, \varphi_x, \varphi_s, \varphi_d, \lambda_z, \lambda_x]$.

Second, I use the median posterior estimates from the first block, together with calibrated parameters, in order to see the asset pricing implications of the model. Here I use two different methods: A log-linear approximation, following Bansal and Yaron (2004); the local projection method, as proposed in Pohl, Schmedders and Wilms (2018). They show that log-linearization may lead to large, economically significant errors for model-implied returns, given the intrinsically nonlinear nature of long-run risks models. In section 4.1 I show results that corroborate this claim.

3.2.1. Bayesian Estimation

Following recent strands in the long-run risks literature, such as Schorfheide, Song and Yaron (2018), Fulop et al (2022), I employ a particle MCMC algorithm to identify long-run risks and estimate the parameters that govern the dynamics of the economy. In this section I give an informal account of the methodology I used to estimate the posterior distributions of parameters. A more complete – and technical – exposition is developed in Appendix B.

For T time periods, I denote all observations as $y_{1:T} = \{\Delta c_t, \Delta d_t, r_t, r_{f,t}, \Delta e_t\}_{t=1}^{T}$ [4] and the latent states as $w_{1:T} = \{\Delta z_t, x_t, \sigma_t^2\}_{t=1}^{T}$. [5] The objective is to compute the joint posterior distribution of parameters and latent states, $p\left(\Theta, w_{1:T} \mid y_{1:T}\right)$, which can be decomposed into:

$$p\left(\Theta, w_{1:T} \mid y_{1:T}\right) = p\left(w_{1:T} \mid \Theta, y_{1:T}\right) p\left(\Theta \mid y_{1:T}\right), \tag{3-1}$$

where Θ denotes the parameter set of the long-run risk model. According to Bayes' Rule,

$$p\left(\Theta \mid y_{1:T}\right) \propto p\left(y_{1:T} \mid \Theta\right) p(\Theta). \tag{3-2}$$

In order to characterize the joint posterior, I employ a particle filter to estimate the parameters that characterize the likelihood $p\left(y_{1:T} \mid \Theta\right)$, while approximating the smoothing distribution of latent states $p\left(w_{1:T} \mid \Theta, y_{1:T}\right)$.

The particle filter draws samples (called particles) from a prior distribution. Each particle has a likelihood weight assigned to it, representing the probability of that particle being sampled from the probability density function. Estimates for the likelihood and smoothing distribution are empirical distributions, which are found by combining sample observations and pondering them by their weights.

The main parameters of interest are ρ_x and ρ_z. The first parameter gives the persistence of long-run shocks in this economy. The largest its value, the more relevant long-run risks will be for asset pricing dynamics. Ideally, its value

[4] There are five observed variables: the consumption growth rates (Δc_t), the dividend growth rates (Δd_t), the market returns (r_t), the risk-free returns ($r_{f,t}$) and the exchange rate depreciation (Δe_t).

[5] There are three state variables for each country: the short-run business cycle fluctuation z_t; the long-run risks component x_t and the stochastic volatility component σ_t^2.

should be close to 1. ρ_z gives the the persistence of short-run business cycle shocks. One should expect to find a much smaller coefficient for those.

Furthermore, one is interested in assessing if long-run shocks due specifically to Covid are relevant for asset prices. In order to test for this hypothesis, I estimate the model using two subsamples: Jan-2018 to Dec-2019 and Jan-2020 to Nov-2021. Ideally, one should observe that ρ_x increases in the second subsample if long-run risks are particularly relevant to the pandemic.

Moreover, one may extract the estimated latent states for each period. If effects due to Covid are being captured by the econometric methodology, one should see that latent states between countries become more correlated during the pandemic. Therefore estimating the model using a sample with two countries is helping to pin down the effects of Covid, both in the short and long run.

In order to assess if the autarky assumption (as defined in Colacito and Croce (2011)) is reasonable, I estimate the model under two limiting cases: (1) That country dynamics are completely independent; (2) that countries are perfectly integrated, i.e., they are subject to the same shocks. If (1) is the case, one should be able to estimate each country separately and arrive at results for asset pricing dynamics that are better in matching the data comparing to the benchmark. If (2) is the case, one should be able to impose the same shocks estimated for the US, for instance, as the underlying shocks for Brazil, and arrive at the same result as if shocks are allowed to be cross-country correlated. In Section 4.2 I show the results for both procedures. They imply that the benchmark case is more appropriate, giving better estimates for asset pricing moments.

3.2.2. Solving for Asset Prices: Log-Linear and Local Projection Method

In Section 2.4 and Appendix A, I show how to solve for asset prices using a log-linear approximation, introduced first in Bansal and Yaron (2004). The numerical solution for the log-linearized model consists simply of finding a fixed-point solution for the price-consumption and price-dividend ratio in the linearized system of equations. Given those two variables and the model parameters, one may find the remaining variables and asset pricing moments. This is a relatively simple method to implement in practice.

For the remaining of this section, I give an overview of the Local Projection Method, following Pohl, Schmedders and Wilms (2018). A more complete exposition is developed in Appendix C, where I describe its algorithmic implementation.

Projection methods are a general tool to solve functional equations of the form $(\mathcal{G}z)(w) = 0$, where w resides in a (state) space $W \subset \mathbb{R}^l, l \geq 1$, and z is an unknown solution function with domain w, so $z : W \to \mathbb{R}^m$.

The first step of a projection method is to approximate the unknown function z on its domain w using a linear combination of basis functions. For a set $\{\Lambda_k\}_{k\in\{0,1,\ldots,n\}}$ of chosen basis functions, the approximation $\hat{z}$ of z is

$$\hat{z}(w; \boldsymbol{\alpha}) = \sum_{k=0}^{n} \alpha_k \Lambda_k(w), \tag{3-3}$$

where $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \ldots, \alpha_n]$ are unknown coefficients. Replacing the function z in equation (3-3) by its approximation $\hat{z}$, one can define the residual function $\hat{F}(w; \boldsymbol{\alpha})$ as the error in the original equation,

$$\hat{F}(w; \alpha) = (\mathcal{G}\hat{z})(w; \alpha). \tag{3-4}$$

Instead of solving equation (3-3) for the unknown function z, one attempts to choose coefficients $\boldsymbol{\alpha}$ to make the residual $\hat{F}(w; \boldsymbol{\alpha})$ zero. Note that instead of finding an infinite-dimensional vector space, the problem is reduced to looking for a vector in $\mathbb{R}^{n+1}$.

This problem is unlikely to have an exact solution, so the second central step of a projection method is to impose certain conditions on the residual function, the so-called "projection" conditions, to make the problem solvable. In other words, the purpose of the projection conditions is to establish a set of requirements that the coefficients α must satisfy. For a formulation of the projection conditions, define a "weight function" (term) $\lambda(w)$ and a set of "test" functions $\{g_k(w)\}_{k=0}^{n}$. Then, define an inner product between the residual function $\hat{F}$ and the test function g_k,

$$\int_W \hat{F}(w; \boldsymbol{\alpha}) g_k(w) \lambda(w) dw. \tag{3-5}$$

This inner product induces a norm on the function space w. Natural restrictions for the coefficient vector α are now the projection conditions,

$$\int_W \hat{F}(w; \boldsymbol{\alpha}) g_k(w) \lambda(w) dw = 0, k = 0, 1, \ldots, n. \tag{3-6}$$

In this paper, I use the collocation method. In it, one chooses $n+1$ distinct nodes in the domain, $\{w_k\}_{k=0}^{n}$, and uses Dirac delta functions as the test functions, $g_k = \delta(w - w_k)$. With a weight term $\lambda(w) \equiv 1$, the projection conditions (3-6) simplify to

$$\hat{F}(w_k; \boldsymbol{\alpha}) = 0, k = 0, 1, \ldots, n. \tag{3-7}$$

Simply put, the collocation method determines the coefficients in the approximation (3-3) by solving the square system (3-7) of nonlinear equations.

For the purposes of this paper, I apply the projection method twice. In the first step, I approximate the log price-consumption ratio $\hat{pc}(w)$ by applying the projections on the residual function of the wealth-Euler equation,

$$\begin{aligned}\hat{F}_{pc}(w; \boldsymbol{\alpha}_{pc}) = \int_W [\exp(\frac{1-\gamma}{1-\theta}(\log\beta + (\frac{\theta-\gamma}{1-\theta})\Delta c(w' \mid w) + pc(w') \\ - \log(e^{pc(w)} - 1))) - 1]\mathrm{d}f_w.\end{aligned} \tag{3-8}$$

Once α_{pc} is known, the projections can be applied to the residual function,

$$\begin{aligned}\hat{F}_{pd}(w; \boldsymbol{\alpha}_{pd}) = \int_W \Big[\exp\Big(\frac{1-\gamma}{1-\theta}\log\beta - \frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c(w' \mid w) + \frac{\theta-\gamma}{1-\theta}\hat{r_c}(w' \mid w; \boldsymbol{\alpha_{pc}}) \\ + \log\left(e^{\hat{pd}(w)} + 1\right) - \hat{pd}(w)(w; \boldsymbol{\alpha}_{pd}) + \Delta d(w' \mid w)\Big) - 1\Big]\mathrm{d}f_w\end{aligned} \tag{3-9}$$

to solve for the price-dividend ratio $\hat{pd}(w; \alpha_{pd})$ of any asset. Given those, we may find returns for any asset using the approximate returns formula:

$$r(w' \mid w) = \ln\left(e^{pd(w')} + 1\right) - pd(w) + \Delta d(w' \mid w). \tag{3-10}$$

4
Results

4.1 Main Results

The prior distributions for each parameter and posterior estimates are found in Tables 4.1 and 4.2. For most variables, I chose a normal distribution as a prior, with exceptions given to parameters that have a bounded support, for which I used either a uniform or truncated normal distribution. I chose uninformative priors for the estimation procedure, in order not to bias the results towards a particular desirable conclusion.

In Tables 4.1 and 4.2, the first parameter block gives the persistence of state variables; the second, moments for the consumption and dividend process, plus the fist moment of the stochastic volatility; the third, the estimated variance for each shock in the model; the fourth, the mapping of state variables into the dividend process.

Noticeably, long-run risks matter for consumption and therefore asset pricing dynamics in this model: The estimated coefficients for ρ_x are large for both countries: 0.795 for the US and 0.914 for Brazil. As noticed in Bansal and Yaron (2004), long-run risks are as important to asset pricing behavior as large those coefficients are.

Table 4.1: Estimated Parameters: Full model on Brazilian data (Jan-2018 to Nov-2021)

Parameter	Prior Distribution	Median (posterior)	Std (Posterior)
ρ_z	Uniform(0,1)	0.288	0.094
ρ_x	Uniform(0,1)	0.914	0.068
ρ_s	Uniform(0,1)	0.801	0.047
μ_c	Normal(1, 0.5)	-0.009	0.003
μ_d	Normal(1, 0.5)	0.001	0.001
σ	Tr. Normal(1, 0.5)	0.008	0.002
φ_z	Tr. Normal(1, 0.5)	0.127	0.043
φ_x	Tr. Normal(1, 0.5)	0.061	0.009
φ_s	Tr. Normal(1, 0.5)	0.012	0.003
φ_d	Tr. Normal(1, 0.5)	0.084	0.021
λ_z	Normal(2, 1)	4.873	0.133
λ_x	Normal(2, 1)	4.914	0.224

Notes: Here I present the prior distribution and posterior distribution for each non-calibrated parameter in the LRR model. Results are the median of the posterior distribution and their estimated standard deviation. The first parameter block gives the persistence of state variables; the second, moments for the consumption and dividend process, plus the fist moment of the stochastic volatility; the third, the estimated variance for each shock in the model; the fourth, the influence of state variables in the dividend process.

The short-run business cycle shocks on the other hand are much less persistent: ρ_z has a estimated value of 0.243 for the US, while Brazil has an estimate of 0.288. Notice that this was not imposed by prior distributions. I used the same neutral prior for both processes, and the Bayesian methodology separated persistent long-run risks and transitory short-run business fluctuations.

Table 4.2: Estimated Parameters: Full model on US data (Jan-2018 to Nov-2021)

Parameter	Prior Distribution	Median (posterior)	Std (Posterior)
ρ_z	Uniform(0,1)	0.243	0.083
ρ_x	Uniform(0,1)	0.795	0.097
ρ_s	Uniform(0,1)	0.844	0.052
μ_c	Normal(1, 0.5)	0.002	0.001
μ_d	Normal(1, 0.5)	0.001	0.001
σ	Tr. Normal(1, 0.5)	0.003	0.001
φ_z	Tr. Normal(1, 0.5)	0.086	0.037
φ_x	Tr. Normal(1, 0.5)	0.044	0.012
φ_s	Tr. Normal(1, 0.5)	0.008	0.002
φ_d	Tr. Normal(1, 0.5)	0.053	0.013
λ_z	Normal(2, 1)	2.966	0.298
λ_x	Normal(2, 1)	2.754	0.226

Notes: Here I present the prior distribution and posterior distribution for each non-calibrated parameter in the LRR model. Results are the median of the posterior distribution and their estimated standard deviation. The first parameter block gives the persistence of state variables; the second, moments for the consumption and dividend process, plus the fist moment of the stochastic volatility; the third, the estimated variance for each shock in the model; the fourth, the influence of state variables in the dividend process.

The stochastic volatility component also appears as an important ingredient for determining asset pricing behavior: ρ_s is estimated as 0.844 in the US, 0.801 in Brazil. While stochastic volatility seems to be slightly more persistent in the US, it has a much smaller median compared to Brazil.

In this particular sample, both countries show a small estimated dividend growth (μ_d). However, Brazil has a negative median monthly consumption growth (μ_c). This should not be entirely surprising. Even before Covid, Brazil had a period of low economic growth and a stagnant economy. While there's a recovery in 2021, it makes sense that the country should present such a result given sample. Moreover, on Table 4.3 I show that Brazil had a negative risk premia, which is both consistent with the literature on risk premia in Brazil and the fact that consumption had such a small estimate in sample.

Brazil also has noticeably higher estimates for λ_z and λ_x. These parameters may be interpreted as financial leverage, and thus are entirely consistent with studies using international data (see Kalemli-Ozcan, Sorensen and Yesiltas (2012), for instance). Notice that these variables also are the sensibility of dividends to state variables in each country, meaning that dividends are more sensitive both to short-run and long-run shocks in Brazil comparing to the USA.

I present the asset pricing implications of the model on Tables 4.3 and 4.4. We can see that the proposed model is able to match quite closely observed moments in returns and depreciation rate data. My results also give further evidence to the necessity of applying solution methods that are able to handle nonlinearities: The log-linear approximation gives estimations to the equity risk premia (and its variance) that are much smaller compared to data. Noticeably, the SDF of the United States does a better job: It matches more closely the mean empirical Sharpe Ratio. The stochastic factors combined are able to match Δe reasonably well. The error, however, is larger than with the pricing of other asset prices: The pricing errors seem to combine rather than cancel each other.

Table 4.3: Model Comparison: Asset Pricing Statistics in Brazil (Jan-2018 to Nov-2021)

Statistic	Data	Local Projection Method	Log-Linear Approximation
mean r	0.0022	0.0020	0.0017
mean r_f	0.0038	0.0042	0.0047
mean $r - r_f$	–0.0017	–0.0021	–0.0029
$\sigma(r)$	0.0055	0.0049	0.0042
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0060	0.0052	0.0044
Sharpe ratio	–0.2777	–0.4038	–0.6591
mean Δe	0.0399	0.0317	0.0294
$\sigma(\Delta e)$	0.3663	0.3226	0.2883

Notes: Asset pricing statistics are presented as monthly average returns and variances. Local Projection Method refers to results using this method, described in Appendix C. Log-Linear Approximation refers to results using this method, described in Section 2.4 and Appendix A. Exchange rates are mean difference in logs and its variance.

Most studies of asset returns in Brazil present an average negative equity premium. Carvalho and Oliveira Santos (2020), for instance, find an annual mean equity premium of -1.75% during the period of 1996-2017 using the Ibovespa index returns. As shown in Table 4.3, Brazil exhibits a negative equity risk premium in this sample.

Although no paper focuses on this particular period in data, we may compare the results in Table 4.4 with other papers in which Bayesian methods are also employed with the Long-Run Risks model. For instance, Schorfheide, Song and Yaron (2018) report on their Table VIII similar results. Naturally, errors are larger in this model, which was estimated on a much smaller dataset from a time-series perspective. In the conclusion I discuss a few future steps which may improve upon current results.

Table 4.4: Model Comparison: Asset Pricing Statistics in USA (Jan-2018 to Nov-2021)

Statistic	Data	Local Projection Method	Log-Linear Approximation
mean r	0.0101	0.0093	0.0085
mean r_f	0.0009	0.0011	0.0014
mean $r - r_f$	0.0091	0.0082	0.0071
$\sigma(r)$	0.0160	0.0121	0.0095
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0162	0.0127	0.0095
Sharpe ratio	0.5664	0.6457	0.7474

Notes: Asset pricing statistics are presented as monthly average returns and variances. Local Projection Method refers to results using this method, described in Appendix C. Log-Linear Approximation refers to results using this method, described in Section 2.4 and Appendix A.

On Appendix D I show the analogous results for Tables 4.3 and 4.4 under two subsamples: Jan-2018 to Dec-2019 and Jan-2020 to Nov-2021. As with the full sample, the model is able to match the market returns, risk-free returns and exchange rate depreciation moments in data.

Finally, in Table 4.5 we see the influence of long-run risks implied by Covid on the estimated model: The latent variables estimated in the pandemic period are much more correlated across countries during this event. Moreover, the estimates for the persistence of the long-run risk become significantly larger during Covid, specially in the US. This gives me confidence that I'm being able to capture the long-run effects of Covid, and that those are a significant influence on asset pricing behavior during the pandemic.

Table 4.5: Long-Run Effects Comparison

Sample	ρ_z^{US}	ρ_x^{US}	ρ_z^{BR}	ρ_x^{BR}	$corr(x^{US}, x^{BR})$	$corr(z^{US}, z^{BR})$
Jan-2018 to Dec-2019	0.232	0.762	0.256	0.848	0.368	-0.318
Jan-2020 to Nov-2021	0.262	0.985	0.324	0.931	0.633	0.112
Jan-2018 to Nov-2021	0.243	0.795	0.288	0.914	0.525	-0.264

Notes: Results are point estimates for the persistence parameters for state variables x and z. $corr(.,.)$ refers to the Pearson correlation coefficient between both variables.

One way of look into these results is to calculate the half-life of a shock given persistence:

$$h_{50\%} = -\frac{\ln(2)}{\ln(|\rho|)}. \tag{4-1}$$

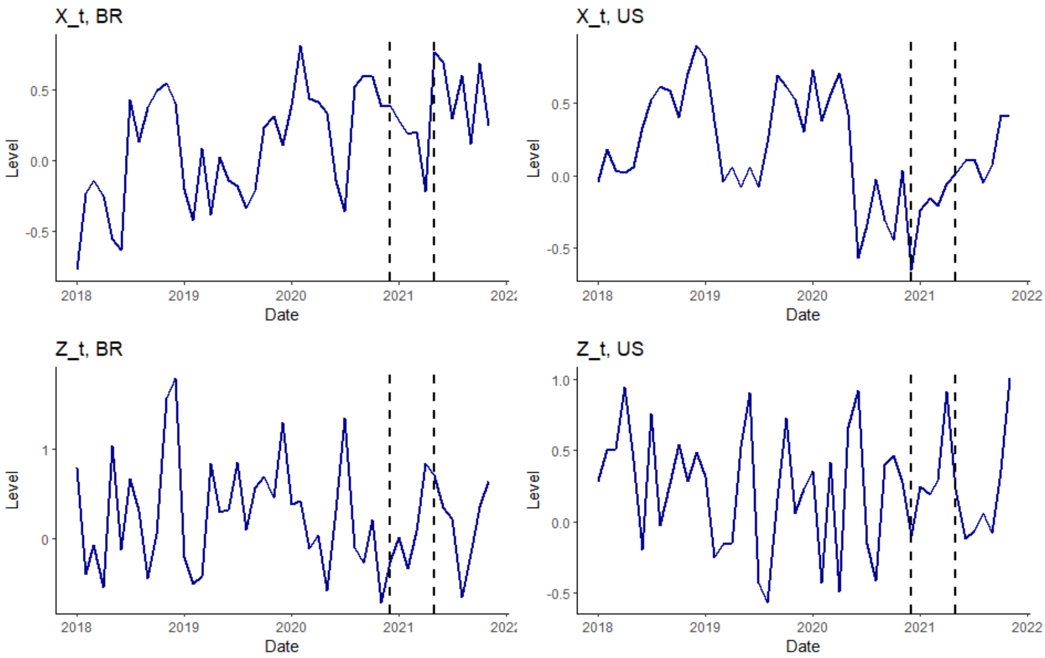
Naturally, shocks relating to z have a half-life that is smaller than one month. z_t^{US} has a half-life of 14 days before Covid and z_t^{BR} has a half-life of 15 days. While these become a little more persistent during Covid, the difference in dissipating time is very small: z_t^{US} value goes to 16 days and z_t^{BR} to 18 days. Shocks relating to x, on the other hand, are much more persistent: ρ_x^{US} of 0.762 means a half-life of 3 months, while 0.985 signifies a half-life of 46 months. ρ_x^{BR} of 0.848 means a half-life of 4 months and 0.931 signifies a half-life of 10 months.

One may also wish to see how long it takes for 95% of a shock to be dissipated. This value can be found plugging ρ in

$$h_{95\%} = -\frac{\ln(20)}{\ln(|\rho|)}. \tag{4-2}$$

Shocks relating to z take about 2 months to dissipate 95% of their value. More accurately, z_t^{US} takes 62 days before the pandemic and 67 after, while z_t^{BR} takes 66 days before and 80 days after the pandemic. Again, the dissipating time is larger for shocks on x: for the US, 95% of their impact is gone in 11 months before the pandemic and 198 months after. For Brazil, the figures are 18 months and 42 months. This means that shocks relating to Covid are much more persistent, and therefore are much more influential in asset pricing behavior.

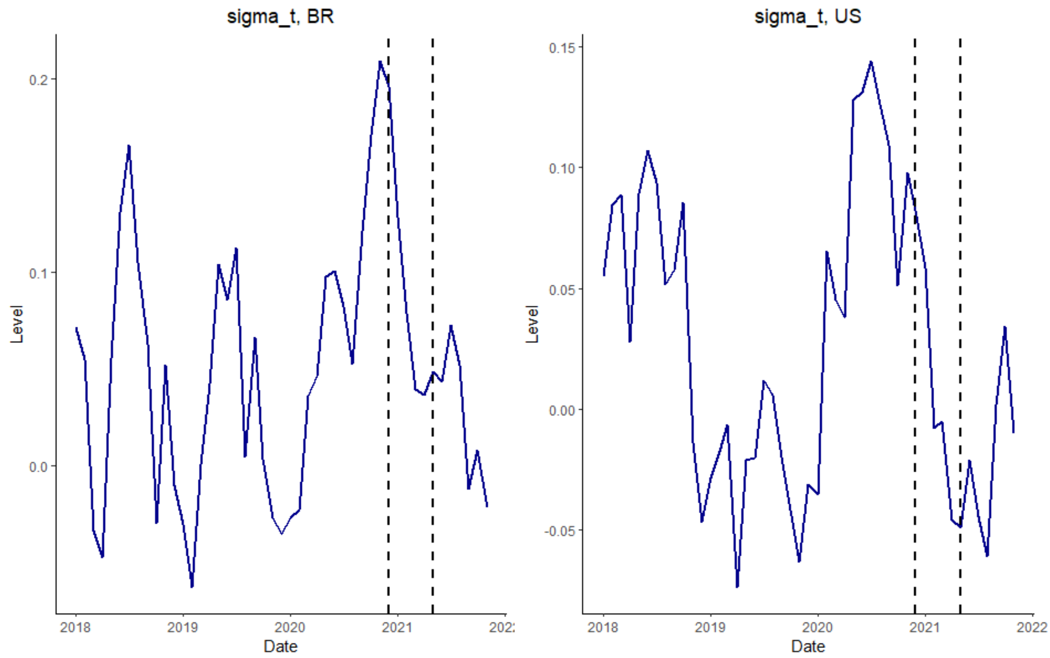
Figure 4.1: State Variables for each Country: x_t and z_t



This Figure plots the estimated state variables between 2018 and 2021. Dashed vertical lines represent announcement dates for Variants of Concern (VOCs).

In Figures 4.1 and 4.2 I plot the estimated state variables for each country. Vertical dashed lines show the dates when Variants of Concern (VOCs) were announced. We may separate the period in three time windows: Before Covid plus its initial shock in March 2020; the period after the first VOC announcement and after the second VOC announcement. Behavior patterns are overall consistent with the fact that they are able to capture COVID shocks: x and z fall heavily at the beginning o the pandemic, while σ increases; that is, consumption suffers both long and short-run negative shocks, while uncertainty increases. Notice that the virus only arrives in Brazil later as compared to the US, so Brazil anticipates the shock. When a variant appears, a similar – yet less pronounced – reaction occurs. Interestingly, the US has an unexpected behavior between the first and second VOC: Its state variable increases in value. We may interpret this fact in the following way: The VOC shocks hit Europe first; Brazil is a small country as compared to Europe, while the US is not. In the US, the state variables are reacting to stimulus measures, and the country follows its own dynamic, while Brazil is much more influenced by events abroad.

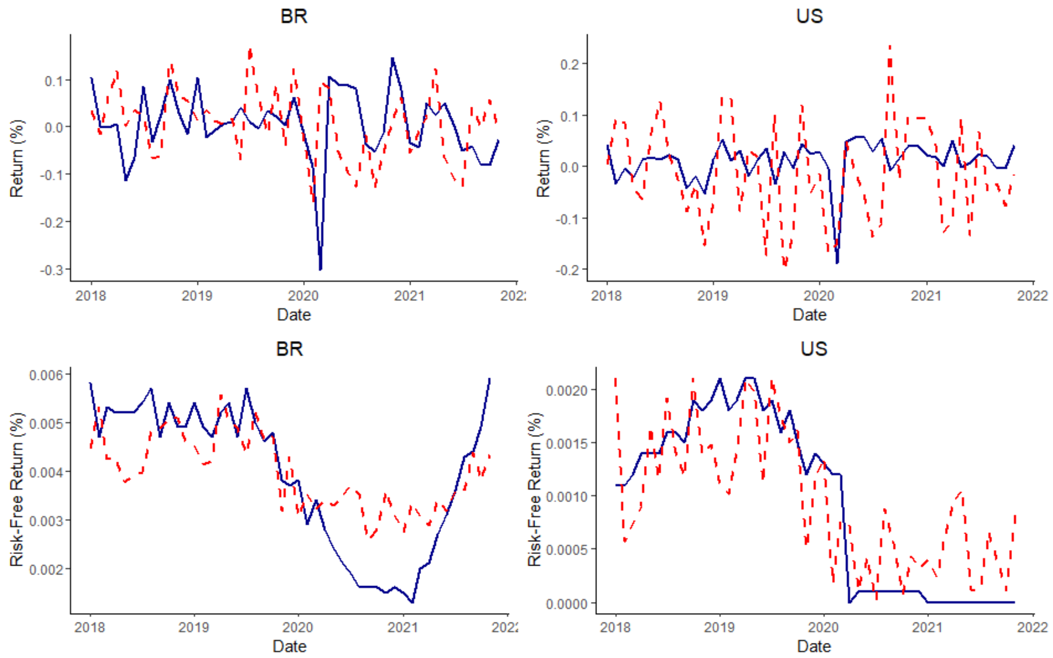
Figure 4.2: State Variables for each Country: σ_t



This Figure plots the estimated state variables between 2018 and 2021. Dashed vertical lines represent announcement dates for Variants of Concern (VOCs).

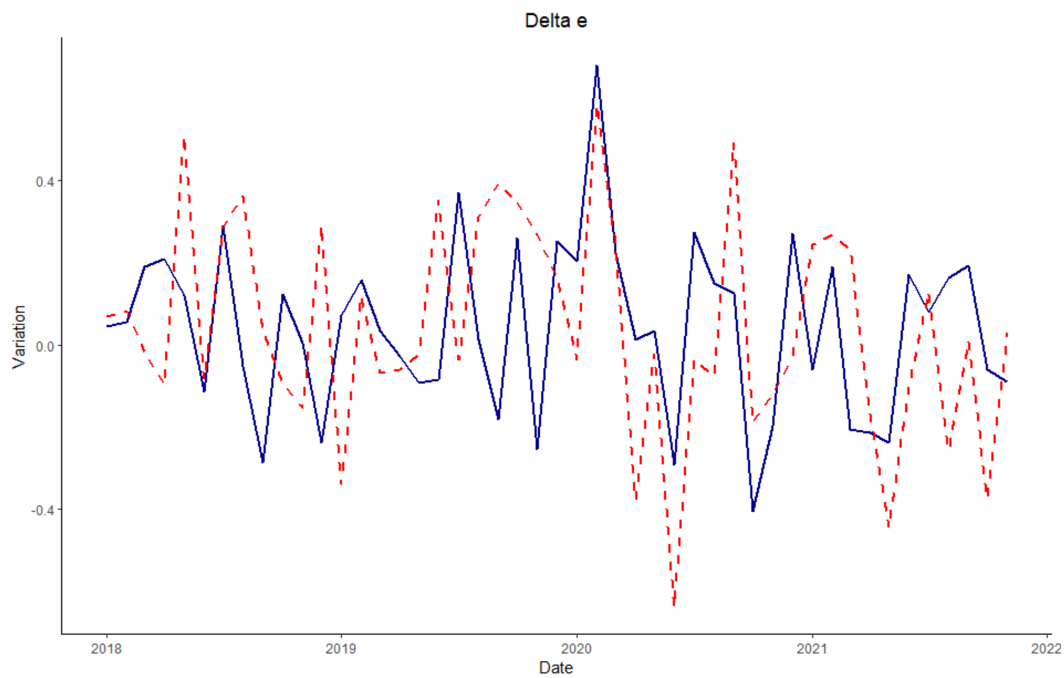
Finally, one may be curious if the model is able to match asset return dynamics. Sadly, the results in Figures 4.3 and 4.4 indicate otherwise. Figure 4.3 shows the realized and model-implied (dashed line) returns and risk-free rate. While the model matches closely asset moments, it has some difficulty in tracking realized trajectories in sample. A similar result holds for exchange rate variation, as shown in Figure 4.4.

Figure 4.3: Fitted Returns: Brazil and US



This Figure plots the realized (blue line) and fitted (dashed red line) returns between 2018 and 2021.

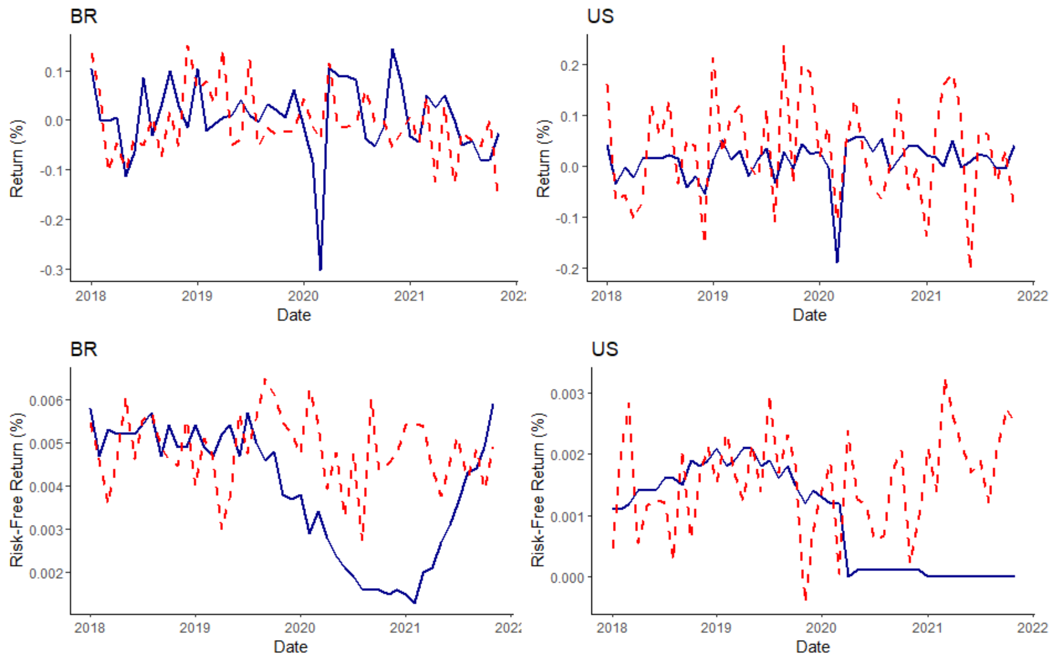
Figure 4.4: Fitted Exchange Rate Variation



This Figure plots the realized (blue line) and fitted (dashed red line) exchange rate variation between 2018 and 2021.

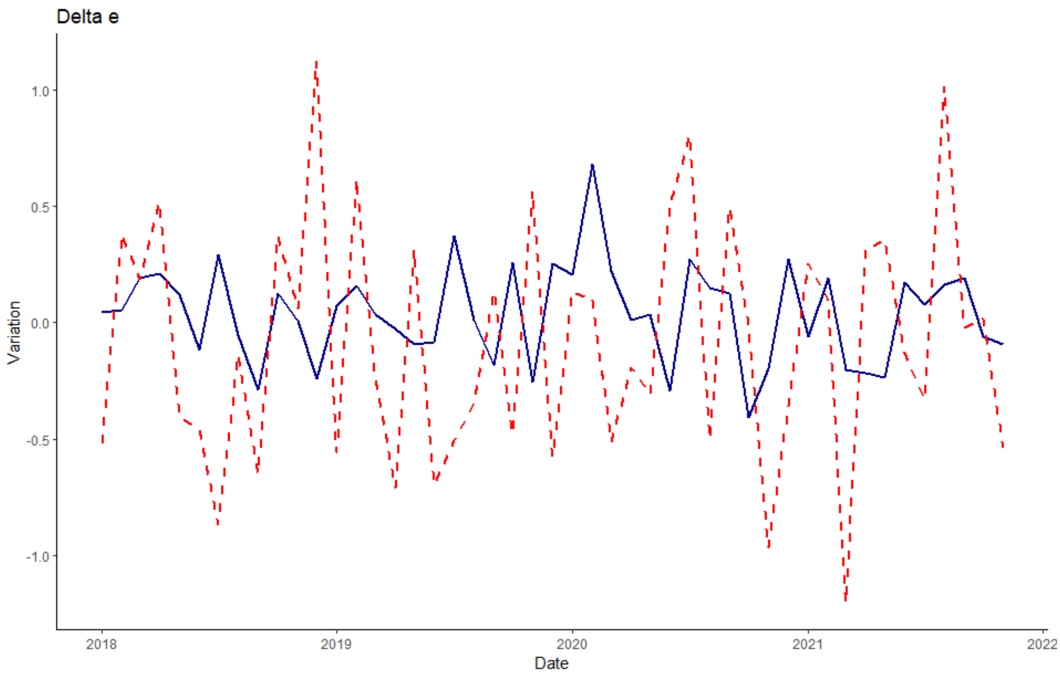
Although this model is not able to track data as closely as one would wish, adding long-run risks helps in this endeavor. Figures 4.5 and 4.6 plot the realized vs. fitted exchange rates in a model with no long-run risks; that is, only z_t affects the consumption and dividend processes. Comparing these figures with 4.3 and 4.4, we can see that the inclusion of x_t helps in tracking the data in sample.

Figure 4.5: Fitted Returns: Brazil and US - Without Long-Run Risks



This Figure plots the realized (blue line) and fitted (dashed red line) returns between 2018 and 2021. In this estimation, x_t does not influence the consumption and dividend processes, only z_t.

Figure 4.6: Fitted Exchange Rate Variation - Without Long-Run Risks



This Figure plots the realized (blue line) and fitted (dashed red line) exchange rate variation between 2018 and 2021. In this estimation, x_t does not influence the consumption and dividend processes, only z_t.

4.2 Country Integration

As an additional exercise, I verify if the assumption that countries are autarkies may interfere with results. For this, first I estimate the model using the first subsample, Jan-2018 to Dec-2019, with each country being estimated separately and both together. For this comparison, I use the local projection method, as it was shown to be more accurate.

Table 4.6: Verifying Autarky: Asset Pricing Statistics in Brazil (Jan-2018 to Dec-2019)

Statistic	Data	Joint Estimation	Separate Estimation
mean r	0.0148	0.0137	0.0140
mean r_f	0.0050	0.0053	0.0063
mean $r - r_f$	0.0098	0.0084	0.0077
$\sigma(r)$	0.0026	0.0022	0.0024
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0026	0.0023	0.0025
Sharpe ratio	3.7827	3.6522	3.0811

Notes: Asset pricing statistics are presented as monthly average returns and variances. Joint Estimation refers to results when both countries are estimated together. These results were found using the Local Projection Method, described in Appendix C.

Table 4.7: Verifying Autarky: Asset Pricing Statistics in USA (Jan-2018 to Dec-2019)

Statistic	Data	Joint Estimation	Separate Estimation
mean r	0.0056	0.0053	0.0048
mean r_f	0.0016	0.0013	0.0014
mean $r - r_f$	0.0040	0.0040	0.0034
$\sigma(r)$	0.0083	0.0075	0.0067
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0083	0.0076	0.0069
Sharpe ratio	0.4802	0.5263	0.4928

Notes: Asset pricing statistics are presented as monthly average returns and variances. Joint Estimation refers to results when both countries are estimated together. These results were found using the Local Projection Method, described in Appendix C.

As one may readily see on Tables 4.6 and 4.7, estimates for asset pricing statistics are not as accurate, as a general thing, when we estimate each country separately, but the difference between using one or both countries is small. This is consistent with two facts: 1) That exploring the heterogeneity of shocks across countries should help to pin down asset pricing behavior; 2) that the autarky assumption, although strong, is not significantly driving the results.

Table 4.8: Are Countries Perfectly Integrated?: Asset Pricing Statistics in Brazil (Jan-2018 to Dec-2019)

Statistic	Data	Baseline Estimation	Perfectly Correlated Shocks
mean r	0.0148	0.0137	0.0085
mean r_f	0.0050	0.0053	0.0034
mean $r - r_f$	0.0098	0.0084	0.0051
$\sigma(r)$	0.0026	0.0022	0.0007
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0026	0.0023	0.0008
Sharpe ratio	3.7827	3.6522	6.3731

Notes: Asset pricing statistics are presented as monthly average returns and variances. Baseline Estimation refers to results when shocks are assumed to be cross-country correlated. These results were found using the Local Projection Method, described in Appendix C.

Moreover, one may test if it would be reasonable to assume that countries are perfectly integrated. I impose the same shocks estimated for the US as the underlying shocks for Brazil, and compare asset pricing statistics with the baseline case, where I allow for shocks to be cross-country correlated. As one may see in Table 4.8, the estimates are a great deal worse in this particular case, thus strongly rejecting that one could assume perfect integration.

5
Conclusion

In this thesis, I contribute to the macro-based asset pricing literature developed around Covid. I show that, while previous explanations of asset pricing behavior during Covid have been unconvincing, it is possible to explain this behavior by relying on a modified long-run risks model. I separate short-term business cycle fluctuations and long-run persistent shocks and show that, by taking the latter into account, one can reasonably explain what happened to asset prices during the period.

The long-run risks model is able to match data moments on stock market returns, risk-free returns, exchange rate depreciation in a monthly sample of two countries, Brazil and the United States. To my knowledge, this is the first paper that applied a consumption based asset pricing model to explain exchange rate movements during Covid.

Moreover, I solve for asset prices using two methods: The standard log-linearization around the steady state and a local projection method, akin to Judd (1992), Judd (1996) and Fulop et al (2022). I corroborate the findings of Pohl, Schmedders and Wilms (2018) by showing that local projection methods, by taking nonlinearities that are intrinsic to long-run risks models into account, produce more accurate estimates of asset returns during the pandemic.

As to further developments of this work, one might be interested in addressing two main points: First, it would be appealing to increase the amount of countries for which I estimate the model. That could strengthen my claim that this asset pricing model is able to capture the stock market behavior during Covid. It is not immediately clear that adding data in the cross-section should improve results, as the number of parameters to be estimated also increases. Notice however that the model has both common and idiosyncratic shocks. I may increase commonalities between countries in order to offset the growth in the number of parameters. Second, one might be uncomfortable in imposing autarky, as it is a strong hypothesis. I have aimed to show that imposing it would not significantly change results. Still, one might prefer to proceed as Dou and Verdelhan (2015), who model the endowment process in a two country economy as containing both a country-specific component and a global one. Although more realistic, this modelling choice has a tractability trade-off, which would become particularly poignant as the number of countries increase.

6
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A
Approximate Log-Linearized Solution

A.1. Stating the Problem

The laws of motion for consumption and dividends in a given country are:[1]

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \Delta z_{t+1} + \sigma_t \epsilon_{c,t+1}, \\
\Delta d_{t+1} &= \mu_d + \lambda_x x_t + \lambda_z \Delta z_{t+1} + \varphi_d \sigma_t \epsilon_{d,t+1}, \\
\Delta z_{t+1} &= \rho_z \Delta z_t + \varphi_z (\epsilon_{z,t+1} - \epsilon_{z,t}), \\
x_t &= \rho_x x_{t-1} + \varphi_x \sigma_t \epsilon_{x,t}, \\
\sigma_t^2 &= \sigma^2 + \rho_s \left(\sigma_{t-1}^2 - \sigma^2\right) + \varphi_s \epsilon_{s,t}, \\
\xi_t &\sim_{\text{i.i.d.}} N(0, I),
\end{aligned} \tag{A-1}$$

where $\xi_t = \left[\epsilon_{c,t}, \epsilon_{d,t}, \epsilon_{x,t}, \epsilon_{z,t}, \epsilon_{s,t}, \epsilon^*_{c,t}, \epsilon^*_{d,t}, \epsilon^*_{x,t}, \epsilon^*_{z,t}, \epsilon^*_{s,t}\right]$ is the vector of shocks in this economy.

The SDF (Stochastic Discount Factor) for this economy is given (in logs) by:

$$m_{t+1} = \frac{1-\gamma}{1-\theta} \log(\beta) - \frac{1-\gamma}{\frac{1}{\theta} - 1} \Delta c_{t+1} + \frac{\theta - \gamma}{1-\theta} r_{c,t+1}. \tag{A-2}$$

Asset prices are derived using the SDF and the asset pricing condition $\mathbb{E}_t \left[M_{t+1} R_{a,t+1}\right] = 1$, so

$$E_t \left[\exp\left(\frac{1-\gamma}{1-\theta} \log(\beta) - \frac{1-\gamma}{\frac{1}{\theta} - 1} \Delta c_{t+1} + \frac{\theta - \gamma}{1-\theta} r_{c,t+1} + r_{a,t+1}\right)\right] = 1 \tag{A-3}$$

for any asset a with (log) return $r_{a,t+1}$.

Following Bansal and Yaron (2004), I first start by solving the special case where $r_{a,t}$ is $r_{c,t}$, the return on the aggregate consumption claim. Then, I solve for the market return r_t and risk-free rate $r_{f,t}$.

[1]Again, I ommit the subscript i from this exposition, since I will assume the same preferences for each country. Notice that since I allow for shocks to be cross-country correlated, including countries in the sample would naturally lead to different estimates

A.2. The Return on the Consumption Claim, $r_{c,t}$

First, I conjecture that the log price-consumption ratio follows $pc_t = A_0 + A_1 x_t + A_2 \Delta z_{t+1} + A_3 \sigma_t^2$. We may substitute the Campbell-Shiller approximation, $r_{c,t+1} = \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}$[2] into the Euler Equation. The, notice that Δc, x, Δz and σ are all conditionally normal, which makes both m_t and $r_{c,t}$ normal. Exploring this fact, we may write the Euler Equation in terms of the state variables x_t, Δz_{t+1} and σ_t. Since the Euler condition must hold for all values of state variables, all terms involving x_t and Δz_{t+1} must satisfy:

$$
\begin{aligned}
-\frac{1-\gamma}{\frac{1}{\theta}-1} x_t + \frac{1-\gamma}{1-\theta}\left[\kappa_1 A_1 \rho_x x_t - A_1 x_t + x_t\right] = 0 \\
-\frac{1-\gamma}{\frac{1}{\theta}-1} \Delta z_{t+1} + \frac{1-\gamma}{1-\theta}\left[\kappa_1 A_2 \rho_z \Delta z_{t+1} - A_2 \Delta z_{t+1} + \Delta z_{t+1}\right] = 0.
\end{aligned}
\tag{A-4}
$$

It follows that

$$
\begin{aligned}
A_1 &= \frac{1-\theta}{1-\kappa_1 \rho_x}, \\
A_2 &= \frac{1-\theta}{1-\kappa_1 \rho_z}.
\end{aligned}
\tag{A-5}
$$

Similarly, we may collect terms and solve for A_3,

$$
\frac{1-\gamma}{1-\theta}\left[\kappa_1 \rho_s A_3 \sigma_t^2 - A_3 \sigma_t^2\right] + 0.5\left[(1-\gamma)^2 + \left(\frac{1-\gamma}{1-\theta} A_1 \kappa_1 \varphi_x\right)^2\right]\sigma_t^2 = 0. \tag{A-6}
$$

Then,

$$
A_3 = \frac{0.5\left[(1-\gamma)^2 + \left(\frac{1-\gamma}{1-\theta} A_1 \kappa_1 \varphi_x\right)^2\right]}{\frac{1-\gamma}{1-\theta}\left(1-\kappa_1 \rho_s\right)}. \tag{A-7}
$$

Given the solution above for pc_t, we may derive the innovation to the return r_c as a function of the evolution of the state variables and the model parameters:

$$
r_{c,t+1} - E_t\left(r_{c,t+1}\right) = \sigma_t \epsilon_{c,t+1} + B_1 \sigma_t e_{x,t+1} + B_2 (e_{z,t+2} - e_{z,t+1}) + A_3 \kappa_1 \varphi_s \epsilon_{s,t+1}, \tag{A-8}
$$

[2]The constants in the Campbell-Shiller approximation are given by $\kappa_1 = \frac{e^{\bar{pc}}}{1+e^{\bar{pc}}}$ and $\kappa_0 = -\log\left((1-\kappa_1)^{1-\kappa_1}\kappa_1^{\kappa_1}\right)$. $\bar{pc} = A_0 + A_1 \bar{x}_t + A_2 \Delta \bar{z}_{t+1} + A_3 \bar{\sigma}_t^2$.

where $B_1 = \kappa_1 A_1 \varphi_x$ and $B_2 = \kappa_1 A_2 \varphi_z$. Additionally, the conditional variance is

$$\mathrm{var}_t\left(r_{c,t+1}\right) = \left(1 + B_1^2\right)\sigma_t^2 + B_2^2 + \left(A_3\kappa_1\right)^2 \varphi_s^2. \tag{A-9}$$

A.3. The Stochastic Discount Factor

Now that we have $r_{c,t+1}$ and Δc_{t+1}, we may (omitting constants) write the SDF in terms of the state variables,

$$m_{t+1} = \frac{1-\gamma}{1-\theta}\log(\beta) - \frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c_{t+1} + \frac{\theta-\gamma}{1-\theta} r_{c,t+1}, \tag{A-10}$$

$$E_t\left[m_{t+1}\right] = m_0 - \theta(x_t + \Delta z_{t+1}) + A_3\left(\kappa_1\rho_s - 1\right)\left(\frac{\theta-\gamma}{1-\theta}\right)\sigma_t^2, \tag{A-11}$$

$$\begin{aligned} m_{t+1} - E_t\left(m_{t+1}\right) =& -\gamma\sigma_t\epsilon_{c,t+1} + \left(\frac{\theta-\gamma}{1-\theta}\right)\left(A_1\kappa_1\varphi_x\right)\sigma_t e_{x,t+1} \\ &+ \left(\frac{\theta-\gamma}{1-\theta}\right)\left(A_2\kappa_1\varphi_z\right)\left(e_{z,t+2} - e_{z,t+1}\right) \\ &+ \left(\frac{\theta-\gamma}{1-\theta}\right)A_3\kappa_1\varphi_s\epsilon_{s,t+1}. \end{aligned} \tag{A-12}$$

Notice that each right hand term accounts for the market price of risk for each source of risk. Therefore, if we define the market prices of risk as $\zeta_c = -\gamma$; $\zeta_x = -(\frac{\theta-\gamma}{1-\theta})\left(A_1\kappa_1\varphi_x\right)$; $\zeta_z = -(\frac{\theta-\gamma}{1-\theta})\left(A_2\kappa_1\varphi_z\right)$ and $\zeta_s = -(\frac{\theta-\gamma}{1-\theta})A_3\kappa_1$, we may write Equation (A-12) as

$$\begin{aligned} m_{t+1} - E_t\left(m_{t+1}\right) =& \zeta_c\sigma_t\epsilon_{c,t+1} - \zeta_x\sigma_t e_{x,t+1} - \zeta_z(e_{z,t+2} - e_{z,t+1}) \\ &- \zeta_s\varphi_s\epsilon_{s,t+1}. \end{aligned} \tag{A-13}$$

A.4. The Market Return, r_t

The risk premium for any asset is determined by the conditional covariance between the return and m_{t+1}. Therefore, $E_t\left(r_{t+1} - r_{f,t}\right) = -\mathrm{cov}_t\left[m_{t+1} - E_t\left(m_{t+1}\right), r_{t+1} - E_t\left(r_{t+1}\right)\right] - 0.5\,\mathrm{var}_t\left(r_{t+1}\right)$. Now, I show the innovation in the market return. The price-dividend ratio is $pd_t = A_{0,pd} + A_{1,pd}x_t + A_{2,pd}\Delta z_{t+1} + A_{3,pd}\sigma_t^2$. Then,

$$
\begin{aligned}
r_{t+1} &= \Delta d_{t+1} + \kappa_1 A_{1,pd} x_{t+1} - A_{1,pd} x_t + \kappa_1 A_{2,pd} \Delta z_{t+2} \\
&\quad - A_{2,pd} \Delta z_{t+1} + \kappa_{1,pd} A_{3,pd} \sigma_{t+1}^2 - A_{3,pd} \sigma_t^2, \\
r_{t+1} - E_t\left(r_{t+1}\right) &= \varphi_d \sigma_t \epsilon_{d,t+1} + \kappa_1 A_{1,pd} \varphi_c \sigma_t \epsilon_{x,t+1} \\
&\quad + \kappa_1 A_{2,pd} \varphi_z (\epsilon_{z,t+2} - \epsilon_{z,t+1}) + \kappa_1 A_{3,pd} \varphi_s \epsilon_{s,t+1} \\
&= \varphi_d \sigma_t \epsilon_{d,t+1} + \beta_{pd,x} \sigma_t \epsilon_{c,t+1} + \beta_{pd,z} (\epsilon_{z,t+2} - \epsilon_{z,t+1}) + \beta_{pd,s} \varphi_s \epsilon_{s,t+1},
\end{aligned}
\tag{A-14}
$$

where $\beta_{pd,x} = \kappa_{1,pd} A_{1,pd} \varphi_c$, $\beta_{pd,z} = \kappa_{1,pd} A_{2,pd} \varphi_z$ and $\beta_{pd,s} = \kappa_{1,pd} A_{3,pd}$. Moreover,

$$
\operatorname{var}_t\left(r_{t+1}\right) = \left(\beta_{pd,x}^2 + \varphi_d^2\right) \sigma_t^2 + \beta_{pd,z}^2 + \beta_{pd,s}^2 \varphi_s^2. \tag{A-15}
$$

Combining the innovations in the market return and the SDF, the expression for the equity premium is

$$
E_t\left(r_{t+1} - r_{f,t}\right) = \beta_{pd,x} \zeta_{pd,x} \sigma_t^2 + \beta_{pd,z} \zeta_{pd,z} + \beta_{pd,s} \zeta_{pd,s} \varphi_s^2 - 0.5 \operatorname{var}_t\left(r_{t+1}\right). \tag{A-16}
$$

To solve for the constants, I use the fact that $E_t\left[\exp\left(m_{t+1} + r_{t+1}\right)\right] = 1$. Collecting terms for x_t and Δz_{t+1}, one may find that

$$
\begin{aligned}
-\theta x + x \kappa_{1,pd} A_{1,pd} \rho_x - A_{1,pd} x + \lambda_x x &= 0, \\
-\theta \Delta z_{t+1} + \Delta z_{t+1} \kappa_{1,pd} A_{2,pd} \rho_z - A_{2,pd} \Delta z_{t+1} + \lambda_z \Delta z_{t+1} &= 0,
\end{aligned}
\tag{A-17}
$$

and those relations imply that

$$
\begin{aligned}
A_{1,pd} &= \frac{\lambda_x - \theta}{1 - \kappa_{1,pd} \rho_x}, \\
A_{2,pd} &= \frac{\lambda_z - \theta}{1 - \kappa_{1,pd} \rho_z}.
\end{aligned}
\tag{A-18}
$$

To solve for $A_{3,pd}$, I leverage on the fact that $\exp\left\{E_t\left(m_{t+1}\right) + E_t\left(r_{t+1}\right) + 0.5 \operatorname{var}_t\left(m_{t+1} + r_{t+1}\right)\right\} = 1$. By collecting the terms with σ_t. First, however, notice that

$$
\begin{aligned}
\operatorname{var}_t\left(m_{t+1} + r_{t+1}\right) = & \operatorname{var}_t[\zeta_c \sigma_t \epsilon_{c,t+1} - \zeta_x \sigma_t e_{x,t+1} - \zeta_z (e_{z,t+2} - e_{z,t+1}) \\
& - \zeta_s \varphi_s \epsilon_{s,t+1} + \varphi_d \sigma_t \epsilon_{d,t+1} + \beta_{pd,x} \sigma_t \\
& \epsilon_{c,t+1} + \beta_{pd,z} (\epsilon_{z,t+2} - \epsilon_{z,t+1}) + \beta_{pd,s} \varphi_s \epsilon_{s,t+1}] \\
& = H_{pd} \sigma_t^2 + \left[-\zeta_{pd,z} + \beta_{pd,z}\right]^2 + \left[-\zeta_{pd,s} + \beta_{pd,s}\right]^2 \varphi_s^2
\end{aligned}
\tag{A-19}
$$

with $H_{pd} = \left[\zeta_{pd,c}^2 + \left(-\zeta_{pd,x} + \beta_{pd,x}\right)^2 + \varphi_d^2\right]$.

Now, collecting terms we arrive at the following restriction:

$$(\frac{\theta-\gamma}{1-\theta})A_3\left(\kappa_1\rho_s-1\right)+A_{3,pd}\left(\kappa_{1,pd}\rho_s-1\right)+0.5H_m=0, \tag{A-20}$$

which implies that

$$A_{3,pd}=\frac{(\frac{\gamma-\theta}{1-\theta})A_3\left(1-\kappa_1\rho_s\right)+0.5H_{pd}}{\left(1-\kappa_{1,pd}\rho_s\right)}. \tag{A-21}$$

Finally, to find the unconditional variance of returns, notice that

$$\begin{aligned} r_{t+1}-E\left(r_{t+1}\right)=&-\theta(x_t+\Delta z_t)+\beta_{pd,x}\sigma_t\epsilon_{x,t+1}+\beta_{pd,z}(\epsilon_{z,t+2}-\epsilon_{z,t+1})\\ &+\varphi_d\sigma_t\epsilon_{d,t+1}+A_{3,pd}\left(\rho_s\kappa_1-1\right)\left[\sigma_t^2-E\left(\sigma_t^2\right)\right]+\beta_{m,s}\varphi_s\epsilon_{s,t+1}. \end{aligned} \tag{A-22}$$

So, the unconditional variance is:

$$\begin{aligned} \operatorname{var}\left(r_t\right)=&\theta^2[\operatorname{var}(x_t)+\operatorname{var}(\Delta z_{t+1})]+\left[\beta_{pd,x}^2+\varphi_d^2\right]\sigma^2\\ &+\beta_{pd,z}^2+\left[A_{3,pd}\left(\rho s\kappa_1-1\right)\right]^2\operatorname{var}\left(\sigma_t^2\right)+\beta_{pd,s}^2\varphi_s^2. \end{aligned} \tag{A-23}$$

A.5. The Risk Free Rate, $r_{f,t}$

Recall that:

$$E_t\left[\exp\left(\frac{1-\gamma}{1-\theta}\log(\beta)-\frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c_{t+1}+\frac{\theta-\gamma}{1-\theta}r_{c,t+1}+r_{a,t+1}\right)\right]=1. \tag{A-24}$$

Substitute r_a for r_f, then

$$\begin{aligned} r_{f,t}=&-\frac{1-\gamma}{1-\theta}\log(\beta)+\frac{1-\gamma}{\frac{1}{\theta}-1}E_t\left[\Delta c_{t+1}\right]+(\frac{\theta-\gamma}{1-\theta})E_t r_{c,t+1}\\ &-\frac{1}{2}\operatorname{var}_t\left[\frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c_{t+1}+(\frac{\theta-\gamma}{1-\theta})r_{c,t+1}\right]. \end{aligned} \tag{A-25}$$

Now, subtract both sides by $(\frac{\theta-\gamma}{1-\theta})r_{f,t}$ and divide by $\frac{1-\gamma}{1-\theta}$ to arrive at:

$$
\begin{aligned}
r_{f,t} =& -\log(\beta) + \theta E_t\left[\Delta c_{t+1}\right] + \frac{(\gamma-\theta)}{1-\gamma} E_t\left[r_{c,t+1} - r_{f,t}\right] \\
& - \frac{1-\theta}{2(1-\gamma)} \operatorname{var}_t \left[\frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c_{t+1} + (\frac{\theta-\gamma}{1-\theta}) r_{c,t+1}\right].
\end{aligned} \tag{A-26}
$$

Note that $\operatorname{var}_t\left[\frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c_{t+1} + (\frac{\theta-\gamma}{1-\theta}) r_{c,t+1}\right] \equiv \operatorname{var}_t\left(m_{t+1}\right)$ and so $\operatorname{var}_t\left(m_{t+1}\right) = \left(\zeta_{pd,c}^2 + \zeta_{pd,x}^2\right)\sigma_t^2 + \zeta_{pd,z}^2 + \zeta_{pd,s}^2\varphi_s^2$. Moreover, the risk premium for $r_{c,t+1}$ is simply given by $E_t\left[r_{c,t+1} - r_{f,t}\right] = -\zeta_c\sigma_t^2 + \zeta_x B_1\sigma_t^2 + \zeta_x B_2 + \kappa_1 A_3 \zeta_s \varphi_s^2 - 0.5\operatorname{var}_t\left(r_{c,t+1}\right)$, the conditional variance between the return and m_{t+1}. Substituting both expressions in (A-26) yields:

$$
\begin{aligned}
E\left(r_{f,t}\right) =& -\log(\beta) + \theta E(\Delta c) + \frac{(\gamma-\theta)}{1-\gamma} E\left[r_{c,t+1} - r_{f,t}\right] \\
& - \frac{1-\theta}{2(1-\gamma)} \left[\left(\zeta_c^2 + \zeta_x^2\right) E\left[\sigma_t^2\right] + \zeta_z^2 + \zeta_s^2\varphi_s^2\right].
\end{aligned} \tag{A-27}
$$

Finally, the unconditional variance of $r_{f,t}$ is

$$
\operatorname{var}\left(r_{f,t}\right) = \theta^2[\operatorname{var}\left(x_t\right) + \operatorname{var}\left(\Delta z_{t+1}\right)] + \left\{\frac{(\gamma-\theta)}{1-\gamma} Q_1 - Q_2 \frac{1-\theta}{2(1-\gamma)}\right\}^2 \operatorname{var}\left(\sigma_t^2\right), \tag{A-28}
$$

where $Q_1 = \left(-\zeta_c + (\frac{\theta-\gamma}{1-\theta}) B_1^2 - 0.5\left(1 + B_1^2\right)\right)$ and $Q_2 = (\zeta_c^2 + \zeta_x^2)$. As noted in Bansal and Yaron (2004), for all practical purposes, the variance of the risk-free rate is determined by the first term.

B
Details on the Econometric Methodology

Following recent strands in the long-run risks literature, such as Schorfheide, Song and Yaron (2018), Fulop et al (2022), I employ a particle MCMC algorithm to identify long-run risks and estimate the parameters that govern the dynamics of the economy.

We can interpret the model in to the framework of nonlinear state-space models. There are three state variables for each country: the short-run business cycle fluctuation z_t; the long-run risks component x_t and the stochastic volatility component σ_t^2.

There are five observed variables: the consumption growth rates (Δc_t), the dividend growth rates (Δd_t), the market returns (r_t), the risk-free returns ($r_{f,t}$) and the exchange rate depreciation (Δe_t). Assuming that market returns, risk-free returns and exchange rate depreciation are collected with measurement errors, the dynamics of market, risk-free returns and exchange rate depreciation for each country are given by:[1]

$$\begin{aligned} r_t &= f\left(\Delta z_t, x_t, \sigma_t^2, \Delta z_{t-1}, x_{t-1}, \sigma_{t-1}^2, \Delta d_t, \Theta\right) + \sigma_m \eta_{m,t}, \\ r_{f,t} &= g\left(\Delta z_t, x_t, \sigma_t^2, \Theta\right) + \sigma_f \eta_{f,t}, \\ \Delta e_t &= h\left(\Delta z_t, x_t, \sigma_t^2, \Theta\right) + \sigma_e \eta_{e,t}. \end{aligned} \tag{B-1}$$

Θ denotes the parameter set of the long-run risk model, r_t, $r_{f,t}$ and Δe_t are the observed market returns, risk-free returns and depreciation rate. σ_m, σ_f and σ_e are the standard deviations of the respective measurement errors, which are assumed to follow independent standard normal distributions.

$f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are functions determining the model-implied market returns, risk-free returns and exchange rate depreciation. These functions are linear when I use the log-linearization method to solve the models, and are highly nonlinear when I use the projection method.

[1] Throughout this exposition, I omit the fact that I have a set of latent variables and observations for each country. I do that only to facilitate the exposition and make the notation less cumbersome.

For T time periods, I denote all observations as $y_{1:T} = \{\Delta c_t, \Delta d_t, r_t, r_{f,t}, \Delta e_t\}_{t=1}^T$ and the latent states as $w_{1:T} = \{\Delta z_t, x_t, \sigma_t^2\}_{t=1}^T$. The objective is to compute the joint posterior distribution of parameters and latent states, $p(\Theta, w_{1:T} \mid y_{1:T})$, which can be decomposed into:

$$p(\Theta, w_{1:T} \mid y_{1:T}) = p(w_{1:T} \mid \Theta, y_{1:T})\, p(\Theta \mid y_{1:T}), \tag{B-2}$$

where $p(w_{1:T} \mid \Theta, y_{1:T})$ solves the state smoothing problem, and $p(\Theta \mid y_{1:T})$ addresses the parameter inference task. Now, I describe how to use SMC methods to implement model estimation.

First, one uses an efficient particle filter that approximates the filtering distribution, providing an unbiased estimate of the likelihood function. For notational convenience, dependence on Θ is suppressed in most of this exposition.

The basic idea is to approximate the filtering distribution $p(w_t \mid y_{1:t})$ with an empirical distribution, denoted as $\hat{p}(w_t \mid y_{1:t})$, supported on a number of particles in the state-space.

Given N samples, $\left\{w_{t-1}^{(i)}; i = 1, 2, \ldots, N\right\}$, approximating the filtering distribution $p(w_{t-1} \mid y_{1:t-1})$ at time $t-1$, the recursion

$$p(w_t \mid y_{1:t}) \propto \int p(y_t \mid w_t, w_{t-1})\, p(w_t \mid w_{t-1})\, p(w_{t-1} \mid y_{1:t-1})\, dw_{t-1} \tag{B-3}$$

prompts the following importance sampling strategy: First, draw samples $\left\{w_t^{(i)}; i = 1, 2, \ldots, N\right\}$ from a known and easy-to-sample proposal transition density, $n_t(w_t \mid w_{t-1}, y_t)$. Second, attach importance weights, α_t, to account for the difference between the target and the proposal, i.e. for $i = 1, 2, \ldots, N$,

$$\alpha_t^{(i)} = \frac{p\left(y_t \mid w_t^{(i)}, w_{t-1}^{(i)}\right) p\left(w_t^{(i)} \mid w_{t-1}^{(i)}\right)}{n_t\left(w_t^{(i)} \mid w_{t-1}^{(i)}, y_t\right)}. \tag{B-4}$$

The normalized weights are given by $A_t^{(i)} = \alpha_t^{(i)} / \sum_{j=1}^N \alpha_t^{(j)}$. Finally, to deal with the particle degeneracy problem, I focus our computational efforts on the most promising particles by resampling from the weighted particle approximation whenever the effective sample size, $ESS_t = 1/\sum_{i=1}^N \left(A_t^{(i)}\right)^2$, is smaller than some prespecified threshold.

Writing $\left\{b_t^{(i)}; i = 1, 2, \ldots, M\right\}$ as the sampled ancestor indices, I obtain an approximation of the filtering distribution $p(w_t \mid y_{1:t})$ with equally weighted samples

$$\hat{p}\left(w_t \mid y_{1:t}\right)=\frac{1}{N} \sum_{i=1}^N \delta_{w_t^{\left(b_t^{(i)}\right)}}\left(w_t\right), \tag{B-5}$$

where $\delta_z(\cdot)$ denotes the Dirac measure centred on z.

Particle filters provide an estimate of the likelihood of the observations:

$$\hat{p}\left(y_{1:t} \mid \Theta\right)=\prod_{l=2}^t \hat{p}\left(y_l \mid y_{1:l-1}, \Theta\right) \hat{p}\left(y_1 \mid \Theta\right), \tag{B-6}$$

where $\hat{p}\left(y_l \mid y_{1:l-1}, \Theta\right)=\frac{1}{N} \sum_{i=1}^N \alpha_l^{(i)}$. The likelihood estimate produced by a particle filter is unbiased (Del Moral (2004)).

I use the particle based algorithm of Fulop et al. (2021), which approximates the optimal proposal transition density, which has the form $n_t^*\left(\Delta z_t, x_t \mid \Delta z_{t-1}, x_{t-1}, y_t\right)=p\left(\Delta z_t, x_t, \mid \Delta z_{t-1}, x_{t-1}, y_t\right)$. This methods combines the Kalman Filter (KF) for the log-linearized model and the unscented Kalman Filter (UKF, as in Li (2011)) to each particle. The algorithm has two steps:

1. Initialization at $t=0$: draw a set of particles $\left\{w_0^{(i)} ; i=1, \ldots, N\right\}$ from the initial distribution $p\left(w_0\right)$ and assign each particle a weight of $1/N$;
2. For time step $t=1, \ldots, T$ and for each particle $i=1, \ldots, N$:
   (a) For log-linearized models, run KF based on $w_{t-1}^{(i)}$ and new observation y_t to obtain mean $\bar{w}_t^{(i)}$ and variance $P_t^{(i)}$;
   (b) For nonlinear models, run UKF based on $w_{t-1}^{(i)}$ and new observation y_t to obtain mean $\bar{w}_t^{(i)}$ and variance $P_t^{(i)}$;
   (c) Sample $w_t^{(i)} \sim N\left(\bar{w}_t^{(i)}, P_t^{(i)}\right)$;
   (d) Update the weight for each particle using Equation (B.4) and compute the normalized weight;
   (e) Resalmple to obtain equally weighted new particles $\left\{w_t^{(i)} ; i=1, \ldots, N\right\}$.

Then, I rely on a SMC sampler to estimate the posterior distribution of the model parameters. According to Bayes' rule, the posterior distribution of model parameters is given by

$$p\left(\Theta \mid y_{1:T}\right) \propto p\left(y_{1:T} \mid \Theta\right) p(\Theta), \tag{B-7}$$

where the first term on the right-hand side is the likelihood and the second term is the prior. Define an auxiliary variable u_t, which includes all random variables generated by a particle filter at time t. These include all N proposed states and ancestor indices from resampling.

Let $\psi\left(u_{1:T} \mid \Theta, y_{1:T}\right)$ denote the corresponding joint distribution of all auxiliary variables given fixed parameters Θ. The extended posterior distribution of model parameters and auxiliary variables is defined as:

$$\tilde{p}\left(\Theta, u_{1:T} \mid y_{1:T}\right) \propto p(\Theta)\hat{p}\left(y_{1:T} \mid \Theta\right)\psi\left(u_{1:T} \mid \Theta, y_{1:T}\right). \tag{B-8}$$

Since the likelihood estimates are unbiased, this extended posterior admits the desired posterior as a marginal distribution:

$$\int \tilde{p}\left(\Theta, u_{1:T} \mid y_{1:T}\right) du_{1:T} = p\left(\Theta \mid y_{1:T}\right). \tag{B-9}$$

Following Duan and Fulop (2015), I construct a sequence of I distributions bridging between the extended prior $\pi_1\left(\Theta, u_{1:T}\right) = p(\Theta)\psi\left(u_{1:T} \mid \Theta, y_{1:T}\right)$ and the extended posterior $\pi_I\left(\Theta, u_{1:T}\right) = \tilde{p}\left(\Theta, u_{1:T} \mid y_{1:T}\right)$ by defining:

$$\begin{aligned} \pi_i\left(\Theta, u_{1:T}\right) &= \frac{\gamma_i\left(\Theta, u_{1:T}\right)}{Z_i} \\ \gamma_i\left(\Theta, u_{1:T}\right) &= p(\Theta)\hat{p}\left(y_{1:T} \mid \Theta\right)^{\xi_i}\psi\left(u_{1:T} \mid \Theta, y_{1:T}\right), \end{aligned} \tag{B-10}$$

where $0 = \xi_1 < \xi_2 < \cdots < \xi_I = 1$ is a tempering sequence and $Z_i = \int \gamma_i\left(\Theta, u_{1:T}\right) d\left(\Theta, u_{1:T}\right)$ is the normalizing constant for the $i = 1, 2, \ldots, I$ distribution. The last distribution of the sequence is exactly the extended posterior, $\tilde{p}\left(\Theta, u_{1:T} \mid y_{1:T}\right)$.

I initialize K equally weighted samples $\left\{\Theta^{(k)}, u_{1:T}^{(k)}\right\}_{k=1}^{K}$ from $\pi_1\left(\Theta, u_{1:T}\right)$ by sampling $\Theta^{(k)} \sim p(\Theta)$ from the prior, and then running a particle filter with M state particles to obtain $u_{1:T}^{(k)} \sim \psi\left(u_{1:T} \mid \Theta, y_{1:T}\right)$ for $k = 1, 2, \ldots, K$.

Given K equally weighted samples $\left\{\Theta^{(k)}, u_{1:T}^{(k)}\right\}_{k=1}^{K}$ approximating the intermediate distribution $\pi_{i-1}\left(\Theta, u_{1:T}\right)$, for $i = 2, 3, \ldots, I$, I move on to approximate $\pi_i\left(\Theta, u_{1:T}\right)$ by weighting each parameter particle $\Theta^{(k)}$ by $s_i^{(k)} = \hat{p}\left(y_{1:T} \mid \Theta^{(k)}\right)^{\xi_i - \xi_{i-1}}$ for $k = 1, 2, \ldots, K$. I also estimate the ratio of normalizing constants Z_i/Z_{i-1} using $\widehat{Z_i/Z_{i-1}} = \frac{1}{K}\sum_{k=1}^{K} s_i^{(K)}$.

Them, I re-sample from the weighted particle approximation to obtain a new set of equally weighted samples $\left\{\Theta^{(k)}, u_{1:T}^{(k)}\right\}_{k=1}^{K}$ that approximate $\pi_i\left(\Theta, u_{1:T}\right)$.

In order to avoid the particle impoverishment problem, I do a rejuvenation step using a particle marginal Metropolis-Hastings (PMMH). For each parameter particle, a PMMH iteration involves proposing a new parameter Θ^*, and accepting this proposal with a Metropolis-Hastings acceptance probability that requires estimating the likelihood $p\left(y_{1:T} \mid \Theta^*\right)$ with a particle filter.

The output of this method gives a set of equally weighted samples $\left\{\Theta^{(k)}, u_{1:T}^{(k)}\right\}_{k=1}^{K}$ approximating the extended posterior $\tilde{p}\left(\Theta, u_{1:T} \mid y_{1:T}\right)$ and the ratio of normalizing constant estimates $\left\{\widehat{Z_i/Z_{i-1}}\right\}_{i=2}^{I}$. Using the output, I can approximate expectations of the form $E[\varphi(\Theta) \mid y_{1:T}]$ using the empirical average $\hat{\varphi} = \frac{1}{K}\sum_{k=1}^{K} \varphi\left(\Theta^{(k)}\right)$ and estimate the marginal likelihood using $\hat{p}\left(y_{1:T}\right) = \prod_{i=2}^{I} \widehat{Z_i/Z_{i-1}}$.

C
Local Projection Method

Following Pohl, Schmedders, and Wilms (2018), Fulop et al (2022), I implement a solution to the model using the collocation projection method (as in Judd (1992)).

Let w be the current state of the economy ($w = \{x, \Delta z, \sigma^2\}$) and w' be the state in the next period. First, I use the projection method to solve the Euler Equation for the price-consumption ratio, *pc*. This solution is such that

$$\mathbb{E}\left[\exp\left(\frac{1-\gamma}{1-\theta}\left(\log\beta + \left(\frac{\theta-\gamma}{1-\theta}\right)\Delta c\left(w' \mid w\right) + pc(w') - \log\left(e^{pc(w)} - 1\right)\right)\right) \mid w\right] = 1, \tag{C-1}$$

and the log-return of the consumption claim is:

$$r_c\left(w' \mid w\right) = pc\left(w'\right) - \ln\left(e^{pc(w)} - 1\right) + \Delta c\left(w' \mid w\right). \tag{C-2}$$

I approximate the solution function by $\hat{pc}(w) = \sum_{k=0}^{n} \alpha_{pc,k}\Lambda_k(w)$, where $\Lambda_k(w), k = 0, \ldots, n$ is a set of (known) basis functions and $\alpha_{pc,k}, k = 0, \ldots, n$ is a set of unknown coefficients, to be determined.

Next, I solve for the price-dividend ratio, *pd*. The Euler Equation gives:

$$\mathbb{E}\left[\exp\left(\frac{1-\gamma}{1-\theta}\log\beta + \left(\frac{1-\gamma}{\frac{1}{\theta}-1}\right)\Delta c\left(w' \mid w\right) + (\frac{\theta-\gamma}{1-\theta})r_c\left(w' \mid w\right) + r\left(w' \mid w\right)\right) \mid w\right] = 1, \tag{C-3}$$

where $r(w' \mid w)$ is such that

$$r\left(w' \mid w\right) = \ln\left(e^{pd(w')} + 1\right) - pd(w) + \Delta d\left(w' \mid w\right). \tag{C-4}$$

As with the price-consumption ratio, the price-dividend is approximated by $\hat{pd}(w) = \sum_{k=0}^{n} \alpha_{pd,k}\Lambda_k(w)$, with $\alpha_{pd,k}, k = 0, \ldots, n$ being a set of unknown coefficients, to be determined.

I solve for the price-consumption and price-dividend ratio using the collocation method; those functions are approximated by using Chebyshev polynomials. The expectations are calculated using Gauss-Hermite quadrature. The collocation approach leads to a square system of nonlinear equations, which can be solved with a standard nonlinear equation solver. Below I describe the algorithm implemented.

Algorithm 1: Solving Asset Pricing Models with Recursive Preferences.

Initialization: Define the state space $W \subset \mathbb{R}^l$; choose the functional forms for $\hat{pc}(w)$ and $\hat{pd}(w)$;

Step 1: Use the wealth-Euler equation together with the approximated log price-consumption ratio $\hat{pc}(w)$ and the definition of the return on the consumption claim to derive the residual function for the return on wealth:

$\hat{F}_{pc}\left(w; \boldsymbol{\alpha}_{pc}\right) =$
$\int_W \left[\exp\left(\frac{1-\gamma}{1-\theta}\left(\log\beta + \left(\frac{\theta-\gamma}{1-\theta}\right)\Delta c\left(w' \mid w\right) + pc(w') - \log\left(e^{pc(w)} - 1\right)\right)\right) - 1\right] \mathrm{d}f_w.$

Compute the unknown solution coefficients $\boldsymbol{\alpha}_{pc}$ by imposing the projections on $\hat{F}_{pc}\left(w; \boldsymbol{\alpha}_{pc}\right)$;

Step 2: Use the solution for the price-consumption ratio $\hat{pc}(w)$ and the Euler equation for an asset together with the approximated log price-dividend ratio $\hat{pd}(w)$ and the definition of the return equation to derive the residual function for an asset:

$\hat{F}_{pd}\left(w; \boldsymbol{\alpha}_{pd}\right) =$
$\int_W \left[\exp\left(\frac{1-\gamma}{1-\theta}\log\beta - \frac{1-\gamma}{\frac{1}{\theta}-1}\Delta c\left(w' \mid w\right) + \frac{\theta-\gamma}{1-\theta}\hat{r}_c\left(w' \mid w; \boldsymbol{\alpha_{pc}}\right)\right.\right.$
$\left.\left. + \log\left(e^{\hat{pd}(w)} + 1\right) - \hat{pd}(w)\left(w; \boldsymbol{\alpha}_{pd}\right) + \Delta d\left(w' \mid w\right)\right) - 1\right] \mathrm{d}f_w$

Compute the unknown solution coefficients $\boldsymbol{\alpha}_{pd}$ by imposing the projections on $\hat{F}_{pd}\left(w; \boldsymbol{\alpha}_{pd}\right)$;

Evaluation: Choose a set of evaluation nodes $\mathbb{W}^e = \left\{w_j^e : 1 \leq j \leq m^e\right\} \subset W$ and compute approximation errors in the residual function of the price-consumption ratio and the residual function of an asset. If the errors do not satisfy a predefined error bound, start over at Initialization and change the number of approximation nodes or the degree of the basis functions;

In the Initialization step, one needs to choose a set of basis functions for the polynomial approximation and a set of nodes. The solution functions $pc(w)$ and $pd(w)$ are approximated by Chebyshev polynomials. These are obtained via the recursive relationship

$$T_0(\xi) = 1, \quad T_1(\xi) = \xi, \quad T_{k+1}(\xi) = 2\xi T_k(\xi) - T_{k-1}(\xi) \tag{C-5}$$

with $T_k : [-1, 1] \to \mathbb{R}$. Since one needs to approximate functions on the domain W and the Chebyshev polynomials are defined on the interval $[-1, 1]$, the argument for the polynomials needs to be transformed. The basis functions for the approximate solutions $\hat{pc}(w)$ and $\hat{pd}(w)$ are given by

$$\Lambda_k(w) = T_k\left(2\left(\frac{w - w_{\min}}{w_{\max} - w_{\min}}\right) - 1\right) \tag{C-6}$$

for $k = 0, 1, \ldots, n$.

The application of a projection method requires a set of nodes, $\mathbb{W} = \{w_j : 0 \leq j \leq m\} \subset W$; I choose the $m+1$ zeros of the Chebyshev polynomial T_{m+1}. These points are called Chebyshev nodes,

$$\xi_j = \cos\left(\frac{2j+1}{2m+2}\pi\right), \quad j = 0, 1, \ldots, m. \tag{C-7}$$

Since all Chebyshev nodes are in the interval $[-1, 1]$ we need to transform them to obtain nodes in the state space W. This transformation is

$$w_j = w_{\min} + \frac{w_{\max} - w_{\min}}{2}\left(1 + \xi_j\right), \quad j = 0, 1, \ldots, m. \tag{C-8}$$

For the Evaluation step I use $m^e >> m$ equally spaced evaluation nodes in W to evaluate the errors in the residual function. In particular, for an asset I compute the root mean squared errors (RMSE) and maximum absolute errors (MAE) in the residual function. These errors are

$$\text{RMSE}_i = \sqrt{\frac{1}{m^e}\sum_{j=1}^{m^e} \hat{F}_{pd}\left(w_j^e \mid \boldsymbol{\alpha}_{pd}\right)^2}, \tag{C-9}$$

$$\text{MAE}_i = \max_{j=1,2,\ldots,m^e} \left|\hat{F}_{pd}\left(w_j^e \mid \boldsymbol{\alpha}_i\right)\right| \tag{C-10}$$

respectively, with

$$w_j^e = w_{\min} + \frac{w_{\max} - w_{\min}}{m^e - 1}(j-1), \quad j = 1, \ldots, m^e. \tag{C-11}$$

D
Additional Results

Table D.1: Model Comparison: Asset Pricing Statistics in Brazil (Jan-2018 to Dec-2019)

Statistic	Data	Local Projection Method	Log-Linear Approximation
mean r	0.0148	0.0137	0.00128
mean r_f	0.0050	0.0053	0.0059
mean $r - r_f$	0.0098	0.0084	0.0069
$\sigma(r)$	0.0026	0.0022	0.0019
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0026	0.0023	0.0020
Sharpe ratio	3.7827	3.6522	3.4544
mean Δe	0.0314	0.0406	0.0484
$\sigma(\Delta e)$	0.2286	0.1981	0.1699

Notes: Asset pricing statistics are presented as monthly average returns and variances. Local Projection Method refers to results using this method, described in Appendix C. Log-Linear Approximation refers to results using this method, described in Section 2.4 and Appendix A. Exchange rates are mean difference in logs and its variance.

Table D.2: Model Comparison: Asset Pricing Statistics in USA (Jan-2018 to Dec-2019)

Statistic	Data	Local Projection Method	Log-Linear Approximation
mean r	0.0056	0.0053	0.0047
mean r_f	0.0016	0.0013	0.0011
mean $r - r_f$	0.0040	0.0040	0.0036
$\sigma(r)$	0.0083	0.0075	0.0065
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0083	0.0076	0.0067
Sharpe ratio	0.4802	0.5263	0.5373

Notes: Asset pricing statistics are presented as monthly average returns and variances. Local Projection Method refers to results using this method, described in Appendix C. Log-Linear Approximation refers to results using this method, described in Section 2.4 and Appendix A.

Table D.3: Model Comparison: Asset Pricing Statistics in Brazil (Jan-2020 to Nov-2021)

Statistic	Data	Local Projection Method	Log-Linear Approximation
mean r	–0.0107	–0.0122	–0.0116
mean r_f	0.0027	0.0024	0.0027
mean $r - r_f$	–0.0135	–0.0146	–0.0143
$\sigma(r)$	0.0085	0.0073	0.0066
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0098	0.0075	0.0069
Sharpe ratio	–1.3714	–1.9467	–2.0724
mean Δe	0.0534	0.0502	0.0484
$\sigma(\Delta e)$	0.5774	0.5232	0.4772

Notes: Asset pricing statistics are presented as monthly average returns and variances. Local Projection Method refers to results using this method, described in Appendix C. Log-Linear Approximation refers to results using this method, described in Section 2.4 and Appendix A. Exchange rates are mean difference in logs and its variance.

Table D.4: Model Comparison: Asset Pricing Statistics in USA (Jan-2020 to Nov-2021)

Statistic	Data	Local Projection Method	Log-Linear Approximation
mean r	0.0134	0.0126	0.0119
mean r_f	0.0002	0.0005	0.0005
mean $r - r_f$	0.0132	0.0121	0.0114
$\sigma(r)$	0.0244	0.0231	0.0202
$\sigma(r_f)$	0.0001	0.0001	0.0001
$\sigma(r - r_f)$	0.0246	0.0233	0.0205
Sharpe ratio	0.5367	0.5193	0.5591

Notes: Asset pricing statistics are presented as monthly average returns and variances. Local Projection Method refers to results using this method, described in Appendix C. Log-Linear Approximation refers to results using this method, described in Section 2.4 and Appendix A.