Pontifícia Universidade Católica
do Rio de Janeiro

Andre Medeiros Sztutman

Informationally efficient markets under rational inattention

Dissertação de Mestrado

Dissertation presented to the Programa de Pós–graduação em Economia of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Economia.

Advisor : Prof. Carlos Viana de Carvalho
Co-Advisor: Prof. Tiago Couto Berriel

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Abstract

Medeiros Sztutman, Andre; Carvalho, Carlos Viana de (Advisor); Berriel, Tiago Couto (Co-Advisor). **Informationally efficient markets under rational inattention**. Rio de Janeiro, 2017. 52p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

We propose a new solution for the Grossman and Stiglitz [1980] paradox. By substituting a rational inattention restriction for their information structure, we show that prices can reflect all the information available without breaking the incentives of market participants to gather information. This model reframes the efficient market hypothesis and reconciles opposing views: prices are fully revealing but only for those who are sufficiently smart. Finally, we develop a method for postulating and solving Walrasian general equilibrium models with rationally inattentive agents circumventing previous tractability assumptions.

Keywords

Rational Inattention; Information Aggregation; Walrasian General Equilibrium; Portfolio Choice;

Resumo

Medeiros Sztutman, Andre; Carvalho, Carlos Viana de; Berriel, Tiago Couto. **Mercados informacionalmente eficientes sob desatenção racional**. Rio de Janeiro, 2017. 52p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Propomos uma nova solução para o paradoxo de Grossman Stiglitz [1980]. Trocando sua estrutura informacional por uma restrição de desatenção racional, nós mostramos que os preços podem refletir toda a informação disponível, sem quebrar os incentivos dos participantes do mercado em processar informação. Esse modelo reformula a hipótese dos mercados eficientes e concilia visões opostas: preços são completamente reveladores, mas apenas para aqueles que são suficientemente espertos. Finalmente, nós desenvolvemos um método para postular e resolver modelos de equilíbrio geral Walrasiano que circunscreve hipóteses simplificadoras anteriores.

Palavras-chave

Desatenção racional; Agregação de informação; Equilíbrio geral Walrasiano; Escolha de portfolio;

Table of contents

List of figures

List of tables

1
Introduction

Can prices fully reflect costly information? In order to reanalyze that question we build a model of costly information processing, that is similar in spirit to (1). In our model, all agents are price takers, prices are fully revealing - they form a one-to-one map with the states of the world - but there are still incentives to gather and analyze information.

This paper adds to a literature that has provided different answers to the Grossman Stiglitz paradox. (2) builds a model where a large trader with monopoly power may decide on a price function that is not fully revealing. (3) builds an auction model that makes explicit how prices are formed. In their model prices in general are partially revealing, but they can be in very special cases fully revealing. Our model is an alternative answer, which abstracts from market power and strategic behavior in auctions. In other words, we do not touch on the question of how prices are explicitly formed. Closer to our model is (4) and (5). (4) builds an alternative version of (1), with an additional intertemporal dimension and a finite number of agents. In this context, he shows that prices can be fully revealing without breaking apart all the incentives to gather information. But his model has the counterfactual implication that irrespective of the number of agents in the economy, only one agent ends up paying for costly information in equilibrium. In our model, every active market participant processes costly information. Contrary to their claim that "a continuum of agents is irreconcilable with information acquisition in a fully revealing financial market equilibrium"(4, p. 468), we show precisely how can that be the case that information acquisition can be made compatible with fully revealing prices in a Walrasian setting.

Our resolution also does not rest on the assumption of signals that must convey bundled information about common and private values as in (5) or (6).

Our results come from modifying (1) informational structure. Instead of making the random asset supply unknown and prices information processed at no cost, we assume that those three variables are all freely available information, but are costly to process. Our novel methodological feature is building a Walrasian general equilibrium model with rational inattention where the information content of prices is not assumed to be automatically processed, without any costs at all.

This gives another answer to Grossman-Stiglitz paradox: when information is costly, prices can still reflect all information. However, it is no trivial task to tell from prices what is the state of the world. For discerning the state of the world, market participants can process information from different sources, including but not restricted to the information content of prices.

Think of asset markets as a chess game. Prices in the chess game are as signals that tell the place of each piece. However, knowing the prices and the position of each piece is not enough to make a good move: it is necessary to think, i.e., process the information available. Alternatively, think of this model as a miniaturized version of an asset market where there are many different assets being traded and traders act on public information. But the amount of public information is so large that one cannot see all the information, even less process it attentively. In this scenario, traders choose what they pay attention to, paying partial attention to each source of randomness. In this story it is emphasized the costly process of seeing the information, while in the first story it is emphasized the costly process of strategically acting on information. In this static model of rationally inattentive agents, both costs are indiscernible, and we can think as the model as expressing both kinds of information processing constraints. [1]

[1] In rational inattention static models there is an equivalence between signals and actions. For discussions of this revelation principle see (7), (8) or (9).

2 Model

We build a model that is close to (1).As in their model, agents have CARA utility, there are a risky and risk-free asset, and the information about the endowment return is costly.

Differently from their model, we assume that the noisy asset supply and the endowment return are discretely distributed instead of multivariate normal. [1] Differently from (1), we treat each variable symmetrically. In their model, making inferences from prices is costless, the endowment return can be known at a cost, and the noisy asset supply cannot be known at any cost. In our model, the cost of information from these three variables is the same, as the agent faces a general mutual information cost as in the rational inattention literature ((10), (11, 8), (12, 13, 14)).

In the next sections, we describe the model in details, in the following order: timing, budget constraints, entropies and mutual information, utility and information choice, equilibrium, distribution of exogenous variables and, lastly, the numerical strategy.

2.1 Timing

There are three periods. In period 1, agents decide on their informational strategies, that is, they choose what variables they will pay attention to. They are not restricted to linear-Gaussian signals (as in (12),(15),(16, 17)), or any specific parametric distribution of uncertainty. In period two, they observe the signals and place their orders of buying and selling the risky security. In period three, intrinsic uncertainty is realized.

2.2 Budget Constraints

As in (1) we assume that each trader is endowed with $\bar{M}$ and $\bar{X}$ units of a risk-free asset and the risky asset respectively.

[1]We also present a version where the noisy asset supply is normally distributed that shares the same results we present for the discrete version. The reason for the discreteness is that it is a natural assumption for solving the model numerically, as will become clearer in the next sections.

His period 1 wealth (before payoffs are realized, and before the portfolio has been placed) is given by:

$$W_1 = P\bar{X} + \bar{M}$$

Where $\bar{X}$ is the agent endowment of the risky asset and $\bar{M}$ is the agent endowment of the risk-free asset. P is the price of the risky asset in terms of the risk-free asset.

His period 2 wealth (before payoffs are realized, and after the portfolio has been placed) is given by:

$$W_2 = PX + M$$

which satisfies the budget constraint:

$$P\bar{X} + \bar{M} = PX + M$$

The agent problem is formulated as a function of $Z = X - \bar{X}$. This choice translates the fact that agents are sending orders to buy or sell the risky asset and those transactions are paid in dollars. Z will be called the agent action, and also will be called the agent signal. This is so, because of the "revelation principle"that holds in this class of information choice model.

His period 3 wealth, finally is given by:

$$W_3 = R_f M + \mu X = Z(\mu - PR_f) + R_f\bar{M} + \mu\bar{X}$$

Where $\mu = \theta + \epsilon$ is the endowment return, which has two components: θ denotes the part of return that can be processed and known at information processing cost an ϵ denotes intrinsic, uninsurable risk. Further, R_f is the return on the riskless asset.

2.3
Entropy and Mutual Information

We follow (15), (18),(17), (10), (8, 11), (19), (20), (12, 13, 14), and (21) in employing the information theoretical concepts of entropy and mutual information. Those quantities are regarded as “answers to fundamental questions” ((22)) in the field of information theory.[2]

Basically, one can see entropy as a measure of uncertainty. For discretely distributed random variables it is defined as:

[2]For a detailed introduction to the field see (23) and (22).

$$H(Z) = \sum_{z} Pr(z) log \left(\frac{1}{Pr(z)} \right)$$

The analog for continuously distributed variables, the differential entropy is defined as:

$$H(Z) = \int_{S} f(z) log \left(\frac{1}{f(z)} \right) dz$$

The mutual information $I(Z;Y)$ is a measure of dependence between two vector of random variables. It is zero if and only if they are independent, it is always nonnegative and it is symmetric. It can be thought as the reduction in the uncertainty about one random variable (say Y) after seeing another random variable (Z). Also, it is independent of the scale of variables (which is a desirable feature absent in applied work common approaches such as postulating costs functions for information on the basis of the sum of variances). Formally, it is given by:

$$I(Z;Y) = H(Z) - H(Z|Y)$$

or, equivalently,

$$I(Z;Y) = H(Z) + H(Y) - H(Z,Y)$$

or,

$$I(Z;Y) = H(Y) - H(Y|Z)$$

This definition can be derived from a variety of axioms, and it is the only measure that satisfies some appealing and intuitive properties. In addition to those already mentioned, the measure does not depend on the order of observations:

$$I(Y;X,Z) = I(Y;X|Z) + I(Y;Z) = I(Y;Z|X) + I(Y;X)$$

If we are willing to adopt a measure of information which has those properties, then we have no choice left but to adopt Shannon mutual information $I(Z;Y)$, and the base on the logarithm is the only arbitrary assumption we can make. When the base adopted is 2, the unit of measure is called bits, when the base is e the unit is nats.

2.4 Utility and Information Choice

As we stated before, each agent has identical CARA utility over period 3 wealth:

$$EU[W_3(Z,Y)] = E\left[-exp\left(-aW_3(Z,Y)\right)\right]$$

Where, a denotes the absolute risk aversion coefficient. We denote the vector of random variables that are not under control of the agent as $Y = (\theta, P, \tilde{X})$. $\tilde{X}$ denotes the random asset supply.

The problem of the agent is to choose an information structure, and, given the information structure and the signal he receives, to choose and action, a buy or sell order, which can take any real value. In other words he is able to design an experiment that provides him with an optimal signal S, but this experiment is bounded by an information constraint. After the signal is realized, the agent will take an action Z(S), which is the optimal action given his knowledge about the signal S. As many models of information choice, this model features a "revelation principle". This means that the choice of signals can be, without loss of generality, restricted to the choice of signals in the form of actions. Informally, any signal S is only optimal if they are in bijection with actions, otherwise they would convey irrelevant information to the decision maker. This implies that any optimal signal S is equivalent to a signal of the form $S = Z$, where Z is the action taken, that is the buy and sell orders. For more formal discussions of this "revelation principle"see for instance (7), (8) or (9). Therefore, we can formulate the rational inattention problem of the agent as choosing conditional distributions f of $Z|Y$ by maximizing expected utility, subject to an information bound. Formally:

$$\max_f EU(Z,Y)$$

$$s.t.\ I(Z;Y) \leq \kappa$$

This formulation of the problem translates the fact that agents are free to choose any information structure they want. They can look at any of the exogenous variables and obtain a signal with an unspecified kind of error. They are not restricted to observing the variable plus a Gaussian noise, not even signals that have information only about each independent component. They can obtain signals that are non-linear combinations of variables plus a noise that has any distributional shape. They can obtain as well direct signals of the state of the world, instead of looking at signals which explicitly reference variables.

A similar formulation consists of choosing conditional distributions f of $Z|Y$ by maximizing expected minus an informational cost that is linear in the mutual information:

$$\max_f EU(Z,Y) - \lambda I(Z;Y)$$

When the information constraint is binding, the two approaches are, in a sense, equivalent since λ can be interpreted as the Lagrange multiplier for the information constraint. This permits employing (24) algorithm for finding a numerical solution to the program above.

2.5 Equilibrium

Defining and solving Walrasian general equilibrium models with rationally inattentive agents is no trivial task. Some additional tractability assumption have been used by the literature. For instance, (20) and (18) assume that agents perfectly observe prices and perfectly process the information contained in prices. [3] This assumption was called by (14) a "schizophrenic compromise".

We show that we can dispense with this compromise by working with the following definitions:

Definition: An agent type is defined as an utility function and the parameters it assumes, prior beliefs about the distribution of exogenous variables and a solution of the rational inattention problem as defined above.

Definition: A replica of an agent type is an iid drawn from any stochastic process that is idiosyncratic. In particular, we assume that information processing is an idiosyncratic process. In other words, replicas face the same distribution of exogenous variables, have the same utility function and decide on the same conditional distribution of signals, but have different draws for idiosyncratic iid processes. [4]

Definition: Given a set of types of agents, denote by r the number of replicas of each type. A Walrasian General Equilibrium with rational inattention is defined as: i) a price function $P(\omega_A)$, where ω_A denotes the aggregate state, ii) for each type $j \in J$ a solution f_j^* of the rational inattention problem, such that:

[3] In the Online Appendix, (18) present a version of their model where the information content of prices is not assumed to be costless processed, generating the interesting and disturbing result that under agents would prefer not to observe prices. However, their model has additional assumptions on the informational structure which we refrain from adopting. For instance, in their model the information cost is linear in the sum of conditional variances of the independent risk factors and prices.

[4] This definition can accommodate asymmetric equilibria by having different types with the same distribution of exogenous variables and utility function but with different solutions for the rational inattention problem if there are many.

$$\lim_{r\to\infty} Pr(\sum_{j\in J}\sum_{i=1}^{r} Z_{i,j} = 0) = 1 \qquad \text{for every aggregate state } \omega_A$$

We need an infinite number of replicas in order for an equilibrium to exist. In a general environment in the appendix, we prove formal versions of the last statements and of Theorem below.

Theorem 2.1 *(Informally stated) Let there be an endowment economy with a finite number of goods and assets, where every agent solves the rational inattention as defined above. With a finite number of agents there is never a price function that clears the markets with probability one for every aggregate state. A large number of agents adds just enough regularity for an equilibrium to exist in great generality.*

Proof: See Appendix, sections A.3 and A.4.

We claim that this definition of equilibrium settles down the question over how to build Walrasian general equilibrium models with rational inattention. The assumption of a large number of agents is a natural one in a Walrasian setting, since it justifies agents being price takers and not acting strategically. Additionally, Theorem 2.1 justifies that an equilibrium exists under a variety of environments and assumptions, not only the one we discuss in greater detail in the rest of the paper.

2.6 Distribution of exogenous variables

In order to close the model, what remains to be done is to postulate a joint distribution for the exogenous variables.

In the next section we first assume, as (1) and much of the literature do, that there is an exogenous noisy supply of the risky asset. This assumption as can be thought as an unmodeled liquidity risk and it could be formalized as wealth risk in a CRRA model.

We keep the structure as simple as possible: let there be four aggregate states in the world. The endowment return θ can be high (θ_H) or low (θ_L) with equal probability and the noisy asset supply (denoted by $\tilde{X}$), independently, can be high ($\tilde{X}_H$) or low ($\tilde{X}_L$) with equal probability. Therefore there are four aggregate states all with 1/4 probability. In each state the random variables assume the following values:

$$\{(\theta_H, \tilde{X}_H), (\theta_H, \tilde{X}_L), (\theta_L, \tilde{X}_H), (\theta_L, \tilde{X}_L)\}.$$

Further, we assume that intrinsic risk ϵ is N(0,σ_ϵ^2) independent of θ.

In the section 4, we build a more tractable and simplified version where the signals and the disturbance are by hypothesis jointly normal. That is, in section 4, the optimal signal is not allowed to be of any distributional form, but is restricted to joint normality. In the Appendix section A.2 we discuss what happens when we dispense with the random asset supply in this set-up. Contrary to (20) conjecture, rational inattention by itself does not generate a noise that substitutes for the exogenous noise assumed by the literature. However, it does generate a multiplicity of equilibrium prices. This is another reason to think of this noise in this class of model as an unmodeled liquidity demand, rather than trade based on weakly informative signals.

In the Appendix section A.7, we propose another structure for the exogenous variables, which leads us a bit further away from (1). The objective is to introduce some elasticity on the noisy supply of the risky asset. This is done by postulating an exogenous risk-free asset supply. This version of the model has the interesting feature that for different levels of the information capacity, agents pay attention to different variables. When the information constraint has a very high shadow cost, only prices and noise receive attention. When agents have a lower shadow cost and higher information capacity they also pay attention to the expected endowment return.

2.7 Numerical strategy

We follow (24) in solving the rational inattention problem numerically and employing a variant of the celebrated (25) algorithm, where there is an added step for optimizing over the support of actions. We then use Christopher Sims' (csolve.m) non-linear solver (quasi-newton with random search) to find a solution for the system of equations that characterize the equilibrium as defined in the Equilibrium section above.

We take a somewhat arbitrary set of parameters, presented in table (2.1). The qualitative results are robust to different choice of parameters, as will be seen in the sections below.

Table 2.1: Parameters for numerical solution

Rf	1	σ_ϵ^2	1
a	0.5	θ_H	1.2
κ	0.5	θ_L	0.8
λ	0.0038	$\tilde{X}_H$	0.2
$\bar{X}$	1	$\tilde{X}_L$	-0.2
$\bar{M}$	1	$Pr(\theta, \tilde{X})$	0.25 (uniform)

It is worth restating the meaning of each parameter. Rf is the risk-free rate of return. a is the absolute risk aversion coefficient. κ is the information capacity of the agent in bits (or the amount of information the agent uses), λ is the shadow cost of information (or the linear cost of information in terms of utility). Fixing κ, λ is an endogenous variable, and fixing λ, κ becomes an endogenous variable. $\bar{X}$ is the agent endowment of the risky asset. $\bar{M}$ is the agent endowment of the risk-free asset. Since utility is CARA those two variables do not affect the results. The risky endowment of the agent ($\bar{X}$) only matters through the aggregate amount of the risky asset the economy has. σ_ϵ^2 is the intrinsic risk of the risky asset. θ_H and θ_L are the values the expected endowment return assumes and $\tilde{X}_H$ and $\tilde{X}_L$ are the values the noisy asset supply assumes. Those two random variables are uniformly distributed, so that $P(\theta, \tilde{X}) = 0.25$ for every pair $(\theta, \tilde{X})$. Lastly, this implies that the entropy of exogenous variables is equal to two bits. An agent with a informational capacity equal to or greater than two bits can see and understand perfectly what is happening in the model.

3
Results

3.1
A counterexample to Grossman and Stiglitz conjecture

We are going to show that our model implies two interesting facts about costly information in Walrasian general equilibrium, which we state in the form of two theorems. The first one is in direct contradiction to (1) conjecture that informationally efficient markets are a logical impossibility.

Theorem 3.1 *Define a vector of prices to be fully revealing if they form a one-to-one map with the aggregate states in the economy. Define information to be costly if* $\lambda > 0$*. Then, that information is costly does not imply that prices cannot be fully revealing.*

The proof of the theorem above is numerical and it is a counterexample. Taking the parameters above, we find the following set of equilibrium prices[1]:

Table 3.1: Equilibrium Prices

	Rational Inattention ($\kappa = 0.5$)	Full Information ($\kappa = 2$)
$P(\theta_H, \tilde{X}_L)$	0.8364	0.8
$P(\theta_H, \tilde{X}_H)$	0.5636	0.6
$P(\theta_L, \tilde{X}_L)$	0.4117	0.4
$P(\theta_L, \tilde{X}_H)$	0.1883	0.2

The relevant information in the table above is not the particular prices for each state, but the fact that they form a one-to-one map with the states, that is, prices are fully revealing.

Full information prices admit a closed form expression, which is derived in the Appendix section A.1: $P = \left(\theta - a\bar{\bar{X}}\sigma_\epsilon^2\right)\frac{1}{R_f}$ where, $\bar{\bar{X}} = \bar{X} + \tilde{X}$. This formula generates the results shown in the third column of table (3.1).

Again, for each state there is a different price. In this sense, prices are fully revealing. By knowing prices and the model structure, one would perfectly deduce what are the aggregate states. By assumption, information is costly,

[1]The full set of routines to generate the results is available in Computational Appendix.

that is $\lambda > 0$. The rational inattention framework highlights that it is costly to make this kind of inference. Information constrained agents will have both incentives to gather and process more information and at the same time will not know everything in spite of the fully revealing character of prices.

Our definition of fully revealing prices does not imply that agents end up knowing everything. Actually, only a very smart agent, with $\kappa \geq 2$, would be able to assimilate all the information only looking at prices. The smarter the agents are, the more information they can extract from prices. However, prices as a source of information can provide all information there is to be provided.

Our second result says that also the converse is not true.

Theorem 3.2 $\lambda = 0$ *does not imply that prices form an one-to-one map with the aggregate states. In other words, that information is costless does not imply that prices are fully revealing.*

Again, the proof is by a counterexample. By carefully choosing parameters, full information prices cease to be fully revealing. Take for instance the numerical calibration presented in 2.7, but let the vector of noisy supplies be given by $(\tilde{X}_H, \tilde{X}_L) = (0.4, -0.4)$, that is, twice as before. Then by applying the formula above, we arrive at full information equilibrium prices that are not fully revealing, as it can be seen in table (3.2).

Table 3.2: Full Information Prices
are not always fully revealing

$P(\theta_H, \tilde{X}_L)$	0.9
$P(\theta_H, \tilde{X}_H)$	0.5
$P(\theta_L, \tilde{X}_L)$	0.5
$P(\theta_L, \tilde{X}_H)$	0.1

Prices reveal whether the economy is in states $(\theta_H, \tilde{X}_L)$ or $(\theta_L, \tilde{X}_H)$ but if $P = 0.5$, one cannot tell from prices whether the economy is in $(\theta_H, \tilde{X}_H)$ or in $(\theta_L, \tilde{X}_L)$.

While, in this model, the case against full information prices being fully revealing is knife-edged, relying on the meticulous choice of the parameters, the case for fully revealing prices under rational inattention is quite general, as will become clear from the comparative statics presented below.

3.2 Comparative Statics

3.2.1 Varying the information cost

Our most important comparative statics concern varying the information cost, since without the information cost our model is just a well-known full information benchmark of (1). For each of the figures plotted below, we solved the model for different values of the information cost (λ), from 0 to 0.1 in steps of 0.001.

Across our numerical explorations we highlight some properties of the solutions.

First, the information cost acts as an amplifier of the exogenous noise. The more information constrained the agents are, the more extreme positions they take in equilibrium, while prices become more extreme and volatile. This is a non-trivial general equilibrium effect, since the partial equilibrium effect of inattention (holding prices fixed) is to become more conservative. This can be seen in figure (1), where the blue solid line plots the support of the actions taken in equilibrium as a function of the information cost under the vector of equilibrium prices and the red dashed line plot the support of actions agents would take if prices were held at the full information benchmark. The red dashed line quickly converges to zero, that is, rising the information costs while keeping the prices fixed leads the agents to stop trading and processing information. The blue solid line diverges, agents take more extreme actions with less precision. This can happen only because of the behavior of prices as plotted in figure (2), where we plot the price for each state as a function of the information cost. As the information cost rises prices are driven away from the full information benchmark in the direction of becoming more volatile. This is necessary for the agents to take more extreme positions in equilibrium. On the other hand, this means that the liquidity demanders pay for the scarce attention of traders.

Since agents are taking more extreme positions the traded volume rises. This can be seen in figure (3). As information gets costlier, the amount of speculative trade rises. That is, the amount of trade based on noisy signals rises, and traders take opposite bets on the market. Market participants "agree to disagree", because it is costly to enrich their information sets until all information they have becomes common knowledge.

Figure (4) highlights the non participation effect we briefly touched above. As the market gets more informed (in the sense of smaller λ), the maximum information cost a single zero-mass agent must have in order to be willing to trade diminishes. This can be seen in the wedge there is between the blue solid line and the red dashed line. In other words, the less informed are pushed away from the trading business, as the market gets more informed.

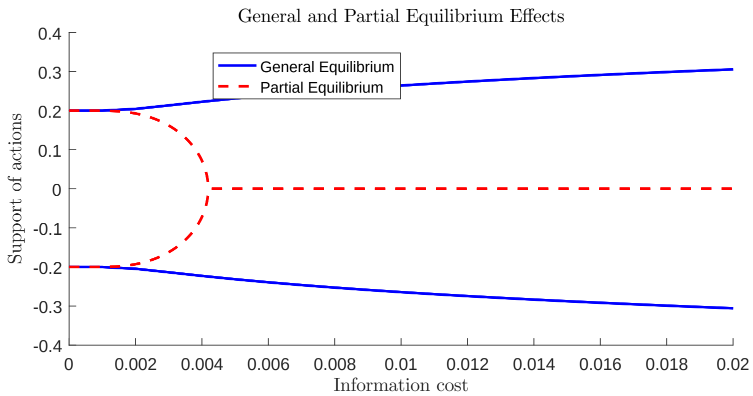

Figure 3.1: General and partial equilibrium effects

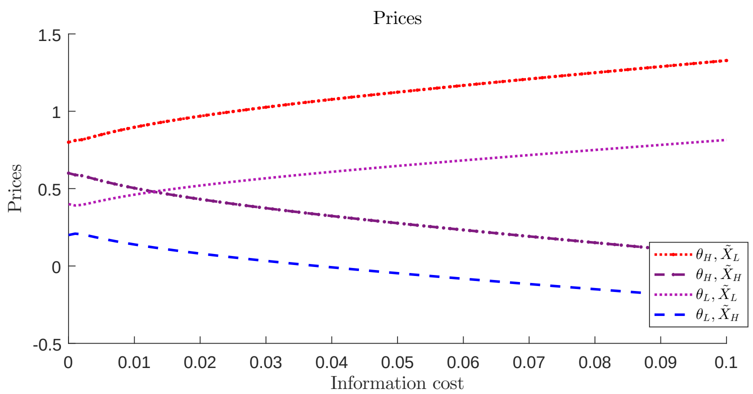

Figure 3.2: Prices

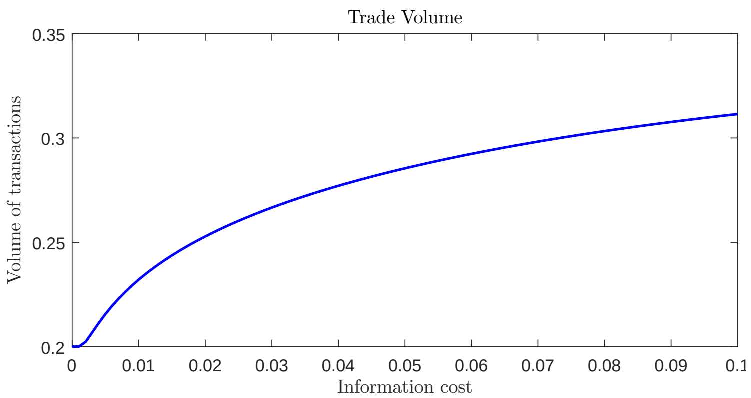

Figure 3.3: Trade volume

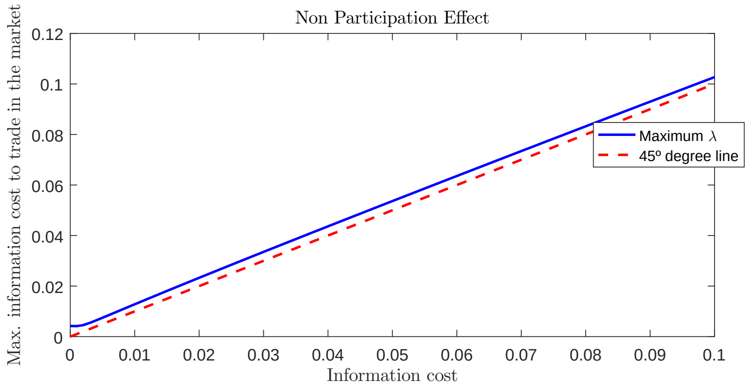

Figure 3.4: Non participation effects

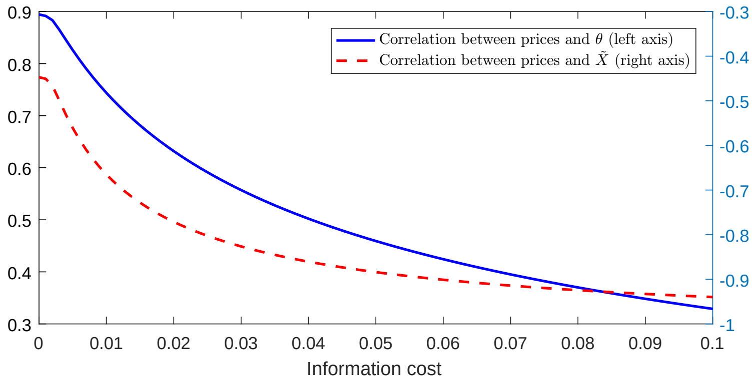

Figure 3.5: Correlation of prices to the liquidity demand and the endowment return

Finally, as information gets costlier, prices become more linearly correlated to noise, and less to the endowment return, as it can be seen in figure (5). However, this does not mean that prices convey more information about noise (or market conditions) and less about the endowment return: prices, almost always, convey all information about those two variables.

From the behavior of prices we can deduce the behavior of other statistics that depend on prices. In the appendix section A.6, we show how the moments of excess returns vary with respect to changes in information cost. In particular, mean excess returns rises as the information becomes costlier and so does the standard deviation. This is indicative that risk coming from imperfect information can explain part of the equity premium. We develop this point further in the section 3.3.

3.2.2 Varying the information capacity

Many of the comparative statics derived for the information cost hold to the information cost, but with the opposite sign. The information cost can be interpreted as the shadow cost of information in equilibrium under the information capacity restriction. However, it could be the case that as people get more informed, the shadow value of information rises. That would happen if information acquisition were a strategic complement. There would be a positive relationship between the marginal value of information and the information capacity of the market. However, as shown above, as people get more informed, the shadow value of information diminishes, indicating that information acquisition is a strategic substitute. Actually, the substitutability effect is so strong that, as people get more informed, expected welfare diminishes.[2] This can be seen on the right panel in figure (6), where we plot the welfare of the rationally inattentive agent as a function of his information capacity. On the left panel, we plot the marginal value of information as a function of information capacity: the marginal value of information smoothly diminishes as the information capacity rises. However, as we said before, this does not comes from the fact that prices reveal more, because prices are almost always fully revealing, in the sense that they form a one-to-one map with the aggregate states.[3] This comes from the fact that there is strategic substitutability in information acquisition and that traders are understanding better the environment.

It would be naive to take this welfare effect as a normative prescription in the sense of prescribing less transparency, since the exogenous liquidity demand is paying for those welfare gains of inattentive traders. Rather, it is more appropriate to understand this welfare effect as part of the competition for supplying liquidity to noise trading, a process by which prices get closer to the full information benchmark.

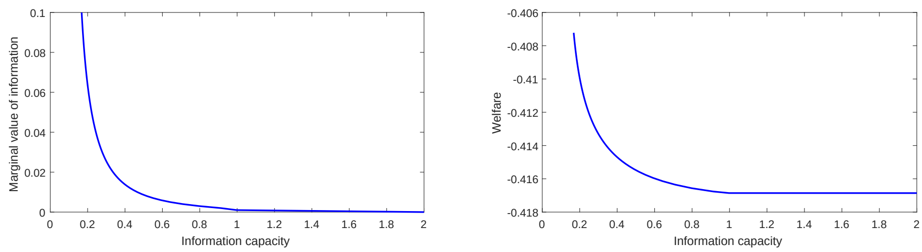

Figure 3.6: Marginal value of information and welfare

Further, is interesting to note that once the information capacity becomes greater than one bit, the marginal value of information goes to zero. In other words, there is irrelevant information in the economy. In the Appendix section

[2]This is true in the κ setup, that is, calculating the expected welfare after the cost of the information has been sunk. Subtracting the information cost from the utility as in the λ setup makes the relationship between welfare and the market information cost negative, but does not change the strategic substitutability effect.

[3]One can argue however that, from the point of view of agents, prices reveal more the less information constrained they are. Only a unconstrained agent, that is a agent with information capacity greater or equal to 2 bits and with $\lambda = 0$, can fully understand the information content of prices.

A.5, we show formally why the information constraint is not binding when $\kappa \in [1, 2]$.

In this simple model, it is the information about the endowment return that becomes irrelevant. The reason for this irrelevance comes from the complete inelasticity of the liquidity demand (noisy asset supply). If we model this liquidity demand as elastic to prices, then the information about the endowment return turns out to be processed in equilibrium. This is shown in Appendix section A.7, in which we build a simple extension of the model.

3.3 Calibration

Our goal in this calibration exercise is to show that imperfect information, in the form of rational inattention, can account for high values of risk aversion necessary to match the data. We do not fully estimate the model, but show that holding the expectation of excess returns fixed, there is a positive relationship between the information capacity and the risk aversion coefficient. We illustrate this finding in figure (7).

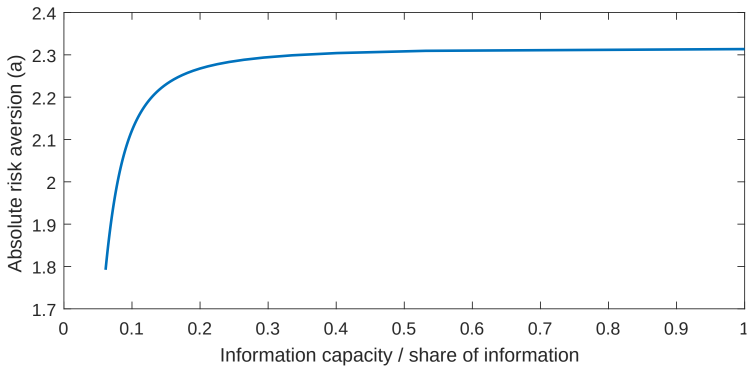

Figure 3.7: Expectation of excess returns are held at 6%

Along the line plotted above, the expectation of excess returns ($E[\theta/P - Rf]$) is held at 6%. At the full information benchmark, the absolute risk aversion coefficient is above 2.3, while by raising the information cost, the absolute risk aversion coefficient steadily diminishes down to less than 1.8, when our numerical algorithm starts to take too long to solve the model. The other parameters are set as in the table below.

Table 3.3: Other parameters

Rf	1.02	σ_ϵ^2	0.023
a	-	θ_H	1.2
κ	-	θ_L	0.8
λ	-	$\tilde{X}_H$	0.2
$\bar{X}$	1	$\tilde{X}_L$	-0.2
$\bar{M}$	1	$Pr(\theta, \tilde{X})$	0.25 (uniform)

That rational inattention can substitute for a high value of risk aversion was already shown by (15). Our novelty here is framing this result in a model where asset prices do not receive a special treatment. Broadly speaking, informational frictions often appear in the literature as one explanation for the equity premium puzzle, see for instance (26).

4
A simplified version with Gaussian signals

We build the simplest model with Gaussian signals that can be compared to the numerical model we developed. As in the numerical example, prices map one-to-one with the aggregate states, in this sense they are fully-revealing. But this property does not harm the incentives to process costly information. Additionally, we can derive many of the properties of the numerical model with closed form expressions.

We assume that the endowment return is made of $\mu = \theta + \epsilon$, where θ is fixed and ϵ is $N(0, \sigma_\epsilon^2)$. The uncertainty we focus on is the uncertainty about the asset supply.

The budget constraint of the agent is:

$$W = X(\theta + \epsilon - PRf) + W_0 Rf$$

We assume that the signal (S) and the noisy asset supply ($\tilde{X}$) are multivariate normal, that is $E(\tilde{X}|S)$ is $N(\mu_{\tilde{X}}, Var(E(\tilde{X}|S)))$. Finally, we assume that agents maximize their expected certainty equivalent wealth as in (20), and (27, 28). That is, they solve at time 1 the problem:

$$\max_{Var(\tilde{X|S})} E(-ln(E(exp(-aW|S)))) \ s.t. \ I(S, \tilde{X}) \leq \kappa$$

At time 2 (right after the signal is observed), the optimal action is found by solving:

$$\max_X E(-exp(-aW(X))|S)$$

We look for an equilibrium price function of the form: $P = A + B\theta + C\tilde{X}$. We do not condition the expectation at time 2 on prices, or on prices and S, but only on S, the optimal multivariate normal signal. This is the Gaussian version of the crucial assumption of the rational inattention theory: all information is available, it is processing that is costly. Nevertheless the optimality of jointly normality of distributions of signals is warranted only under very specific hypothesis on distribution of uncertainty and the utility function ((24)), we follow the applied literature in adopting it because of its simplicity ((20), (17, 16), (15)). Define the auxiliary variable Φ as:

$$\Phi(X) = -a(W(X) - W_0 Rf)$$

Given the signal S, the agent problem can be written as:

$$\underset{X}{max} - E[\Phi(X)|S] - \frac{1}{2}Var[\Phi(X)|S]$$

Solving this problem yields the optimal portfolio holding, conditional on the signal S:

$$X = \frac{((1 - BRf)\theta - CRfE(\tilde{X}|S) - ARf)}{a(\sigma_\epsilon^2 + (CRf)^2 V(\tilde{X}|S))}$$

Now, our equilibrium concept implies that:

$$\lim_{r\to\infty} \sum_r X_r|\tilde{X} = \tilde{X}$$

or equivalently $E(X|\tilde{X}) = \tilde{X}$. Where the expectation on the left hand side is taken with respect to the distribution of signals. Therefore

$$\tilde{X} = \frac{((1 - BRf)\theta - CRf\tilde{X} - ARf)}{a(\sigma_\epsilon^2 + (CRf)^2 V(\tilde{X}|S))}$$

Implying that the price coefficients are $B = 1/Rf$, $A = 0$, and C is given by:

$$C_- = \left(\frac{-1 - (1 - 4a^2\sigma_\epsilon^2 V(\tilde{X}|S))^{1/2}}{2aRfV(\tilde{X}|S)}\right) \; or \; C_+ = \left(\frac{-1 + (1 - 4a^2\sigma_\epsilon^2 V(\tilde{X}|S))^{1/2}}{2aRfV(\tilde{X}|S)}\right)$$

While the full information benchmark is given by: $A_{full} = 0, B_{full} = 1/Rf$ and $C_{full} = \frac{-a\sigma_\epsilon^2}{Rf}$. We must remark that equilibrium does not exist if $a^2\sigma_\epsilon^2 V(\tilde{X}|S) > 1/4$. That means that if risk aversion, the intrinsic risk and the endogenous uncertainty are too high, then there is not an equilibrium. Further notice that $1 - 4a^2\sigma_\epsilon^2 V(\tilde{X}|S) < 1$, which implies that $C < 0$.

The most plausible price functional is the one defined by C_+, which is continuous on uncertainty $(V(\tilde{X}|S))$ and has a limit the full information price function when $V(\tilde{X}|S)$ vanishes. One could use the existence of two price functions consistent with equilibrium to rationalize sudden changes in volatility. However, I argue that the most sensible use of the model is to discard the price functional defined by C_-, since the coefficient C approaches minus infinity as one gets closer to full information: such enormous amount of volatility makes it an implausible discontinuous equilibrium at the neighborhood of the full information benchmark. We discuss below the properties of both equilibria.

Now we check whether there are incentives to acquire information by solving the information acquisition part of the problem. This amounts to taking the solution to X above and plugging it into the utility function $E(-ln(E(exp(\Phi|S))))$.

$$E(exp(\Phi)|S)) = E\left(exp\left(\frac{CRfE(\tilde{X}|S)}{(\sigma_\epsilon^2 + (CRf)^2V(\tilde{X}|S))}(\epsilon - CRf\tilde{X})\right)\Big|S\right)$$

Which is the expectation of a log-normal variable. A straightforward calculation shows:

$$E(-ln(E(exp(\Phi)|S))) = \frac{1}{2}\left(\frac{(CRf)^2(\mu_{\tilde{X}}^2 + \sigma_{\tilde{X}}^2 - V(\tilde{X}|S))}{(\sigma_\epsilon^2 + (CRf)^2V(\tilde{X}|S))}\right)$$

And the rational inattention problem becomes:

$$\max_{V(\tilde{X}|S)}\left(\frac{(CRf)^2(\mu_{\tilde{X}}^2 + \sigma_{\tilde{X}}^2 - V(\tilde{X}|S))}{(\sigma_\epsilon^2 + (CRf)^2V(\tilde{X}|S))}\right)\ s.t.\ 0 \leq I(\tilde{X},S) \leq \kappa$$

The utility is decreasing in $V(\tilde{X}|S)$. Therefore the information constraint is always binding. Now notice that:

$$I(\tilde{X},S) = \frac{1}{2}log_2(2\pi e\sigma_{\tilde{X}}^2) - \frac{1}{2}log_2(2\pi eV(\tilde{X}|S)) = \frac{1}{2}log_2\left(\frac{\sigma_{\tilde{X}}^2}{V(\tilde{X}|S)}\right)$$

Which implies that $V(\tilde{X}|S) = \frac{\sigma_{\tilde{X}}^2}{2^\kappa}$. This finishes solving the model, since we can substitute back the values of $V(\tilde{X}|S)$ on the price coefficient C and on demands. We now discuss some of the model implications.

4.1 Price Volatility

We claim and prove that, in this model, under rational inattention, price volatility is strictly higher than without rational inattention. To see this, notice that the variance of prices is given by $Var(P) = C^2\sigma_{\tilde{X}}^2$ .Provided that the capacity is less than the entropy of $\tilde{X}$, that is $\kappa < \frac{1}{2}log_2(2\pi e\sigma_{\tilde{X}}^2)$, and $4a^2\sigma_\epsilon^2\frac{\sigma_{\tilde{X}}^2}{2^\kappa} < 1$, the following inequalities hold:

$$C_-^2 \geq C_+^2 > C_{full}^2$$

This chain of inequality follow directly from the definitions of C_-, C_+ and C_{full}.

$$C_- = \left(\frac{-1 - (1 - 4a^2\sigma_\epsilon^2 \frac{\sigma_{\tilde{X}}^2}{2^\kappa})^{1/2}}{2aRf \frac{\sigma_{\tilde{X}}^2}{2^\kappa}} \right) , \; C_+ = \left(\frac{-1 + (1 - 4a^2\sigma_\epsilon^2 \frac{\sigma_{\tilde{X}}^2}{2^\kappa})^{1/2}}{2aRf \frac{\sigma_{\tilde{X}}^2}{2^\kappa}} \right)$$

$$and \; C_{full} = \frac{-a\sigma_\epsilon^2}{Rf}$$

Additionally, it is straightforward to see that $\frac{dC_+^2}{d\kappa} < 0$. Which means that for the price functional defined by C_+, as market becomes more informed, prices become less volatile. Rational inattention smoothly amplifies the fundamental volatility $\sigma_{\tilde{X}}^2$, if we restrict ourselves only to this class of equilibrium functions.

The opposite happens for C_-. We have that $\frac{dC_-^2}{d\kappa} > 0$, but price volatility jumps to infinity as we pass from the full information equilibrium to an equilibrium price function of the form $P = \frac{\theta}{Rf} + C_-(\kappa)\tilde{X}$ with a near perfect information capacity, that is, where κ is close to the entropy of exogenous variables. As the capacity κ falls, then prices become less and less volatile, up to the point where $C_- = C_+ = \frac{-2a\sigma_\epsilon^2}{Rf}$, if $4a^2\sigma_\epsilon^2\sigma_{\tilde{X}}^2 \geq 1$. When the information capacity falls below this point, markets shut down, as it becomes too much a risky business.

4.2 Welfare and Strategic Substitutability

The welfare of traders is directly related to the price function parameter C, which as shown before is always negative. As C^2 raises, that is, as prices become more volatile, welfare raises. This coefficient measures how much traders charge for the liquidity provided to noise traders, so one should not take this welfare effect as normative. That is, as a lower C warrants better prices to rationally inattentive traders, liquidity demanders face worse prices for themselves. The expression for welfare of a single agent as a function of C and its own κ_i is:

$$EU(C^2, \kappa_i) = \left(\frac{(CRf)^2(\mu_{\tilde{X}}^2 + \sigma_{\tilde{X}}^2 - \frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}})}{\sigma_\epsilon^2 + (CRf)^2 \frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}}} \right)$$

and one can easily check that:

$$\frac{\partial EU(C^2,\kappa_i)}{\partial C^2} = \frac{(\mu_{\tilde{X}}^2+\sigma_{\tilde{X}}^2-\frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}})\sigma_\epsilon^2}{(\sigma_\epsilon^2+(CRf)^2\frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}})^2} > 0$$

$$\frac{\partial^2 EU(C^2,\kappa_i)}{\partial C^2 \partial\kappa_i} = \frac{ln(2)\sigma_\epsilon^2\frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}}}{\left(\frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}}(CRf)^2+\sigma_\epsilon^2\right)^2}\left(1+\frac{2(CRf)^2\left(\mu_{\tilde{X}}^2+\sigma_{\tilde{X}}^2-\frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}}\right)}{\frac{\sigma_{\tilde{X}}^2}{2^{\kappa_i}}(CRf)^2+\sigma_\epsilon^2}\right) > 0$$

The first derivative shows that welfare is increasing in volatility. The second derivative shows that the marginal value of information is increasing in volatility.

Since C^2 is always greater under rational inattention, welfare is greater for a fully informed atomistic individual. If we restrict ourselves to the equilibria characterized by C_+, then for an atomistic individual with a given capacity, welfare is decreasing in the market information capacity and there is strategic substitutability in information acquisition, as in (1).

The opposite holds in the price functional defined by C_-. There is strategic complementarity in information acquisition. Since the chain of inequalities $C_-^2 \geq C_+^2 > C_{full}^2$ holds, volatility is higher and rational inattentive agents would be better off in this class of equilibrium. [1]

4.3 Non participation

Suppose that instead of a fixed capacity in bits, agents have a fixed linear utility cost for each bit of information. In other words, suppose the agent problem is to maximize $EU - \lambda I$, that is the expected utility minus λ times the mutual information between the distribution of signals and the distribution of uncertainty. Then there is a positive λ that makes the agents decide not to actively trade in the market, and instead take a fixed position which does not vary with the fundamentals. Call the λ that makes the agent to stop processing information λ_N. A simple substitution can show that this λ_N is given by:

$$\lambda_N = \frac{(CRf)^4(\mu_{\tilde{X}}^2+\sigma_{\tilde{X}}^2+\sigma_\epsilon^2)}{((CRf)^2\sigma_{\tilde{X}}^2+\sigma_\epsilon^2)^2}\sigma_{\tilde{X}}^2 ln(2)$$

λ_N is increasing in C^2. Under a low volatile equilibrium, the information cost of a single agent must be lower than in high volatility equilibrium in order for him to actively trade in the asset markets. More volatile markets attract

[1] Notice we haven't allowed for non fundamental volatility, that is switching between C_- and C_+, which is an extension left for future work.

more traders. Restricting ourselves to the equilibrium price functional defined by $P = \frac{\theta}{Rf} + C_+(\kappa)\tilde{X}$, as the market κ rises, λ_N diminishes. That is, as the market gets more informed, it crowds out uninformed traders. The opposite happens if we restrict our attention to $P = \frac{\theta}{Rf} + C_-(\kappa)\tilde{X}$: as the market gets more informed, volatility rises, and it crowds in more uninformed traders.

4.4 Profits

The agents who have greater information capacity achieve higher average profits. Profits for an atomistic individual, holding prices fixed, are given by:

$$E(W(\kappa_i)) = \frac{2^{\kappa_i}(CRf)^2\mu_{\tilde{X}}}{a(\sigma_\epsilon^2 + (CRf)^2\sigma_{\tilde{X}}^2)} + WoRf$$

Which is increasing in κ. In other words, up to the point of full information, higher information capacity translates into higher profits. Additionally, a rational inattention equilibrium makes the rational inattentive agent richer than in a full information equilibrium. Again, as in the welfare analysis, this should not be taken as a normative result since the unmodeled liquidity demand is paying for the wealth of traders.

5
Conclusion

This paper fills a gap by solving rational inattention in Walrasian general equilibrium, without previous tractability assumptions, claimed by (14) to be incompatible with rational inattention modeling.

The model revisits the Grossman-Stiglitz paradox, by showing that prices can form a one-to-one map with the states of the world, but still provide incentives for information gathering. The key in that model is that agents are rationally inattentive over the information contained in prices.

Additionally, in this model there is strategic substitutability in information acquisition and higher information capacity allows agents to achieve higher profits while a high information cost makes agents not to participate in the asset markets. Prices are more volatile than the endowment return and inattention can substitute for a high risk aversion.

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A
Appendix

A.1
Full information prices and demands

Time 2 expected utility is given by

$$EU[W_{3i}|Z_i] = E\left[-exp\left(-aW_{3i}\right)\Big|Z\right]$$

$$= E\left[-exp\left(-a(Z_i(\mu - PR_f) + R_f\bar{M} + \mu\bar{X})\right)\Big|Z_i\right]$$

We assume that $\epsilon|Z_i$ is N(0,σ_ϵ^2) conditionally independent of θ

Therefore we can write:

$$EU[W_{3i}|Z_i] = -E\left[exp\left(-a(Z_i(\theta - PR_f) + R_f\bar{M} + \theta\bar{X})\right)\Big|Z_i\right] * E\left[exp\left(-a(\bar{X} + Z_i)\epsilon\right)\Big|Z_i\right]$$

Then,

$$EU[W_{3i}|Z_i] = -E\left[exp\left(-a(Z_i(\theta - PR_f) + R_f\bar{M} + \theta\bar{X}\right))\Big|Z_i\right] * exp\left(\left(a(\bar{X} + Z_i)\right)^2 \frac{\sigma_\epsilon^2}{2}\right)$$

Suppose θ and P are known. Then the expression above can be written as:

$$-exp\left(-a(Z_i(\theta - PR_f) + R_f\bar{M} + \theta\bar{X})\right) * exp\left(\left(a(\bar{X} + Z_i)\right)^2 \frac{\sigma_\epsilon^2}{2}\right)$$

Taking the log:

$$-\left[\left(-a(Z_i(\theta - PR_f) + R_f\bar{M} + \theta\bar{X}\right) + \left(\left(a(\bar{X} + Z_i)\right)^2 \frac{\sigma_\epsilon^2}{2}\right)\right]$$

maximizing with respect to Z_i

$$Z_i = \frac{(\theta - R_f P)}{a\sigma_\epsilon^2} - \bar{X}_i$$

$$P = \left(\theta - a\bar{\bar{X}}\sigma_\epsilon^2\right)\frac{1}{R_f}$$

where, $\bar{\bar{X}} = \bar{X} + \tilde{X}$.

■

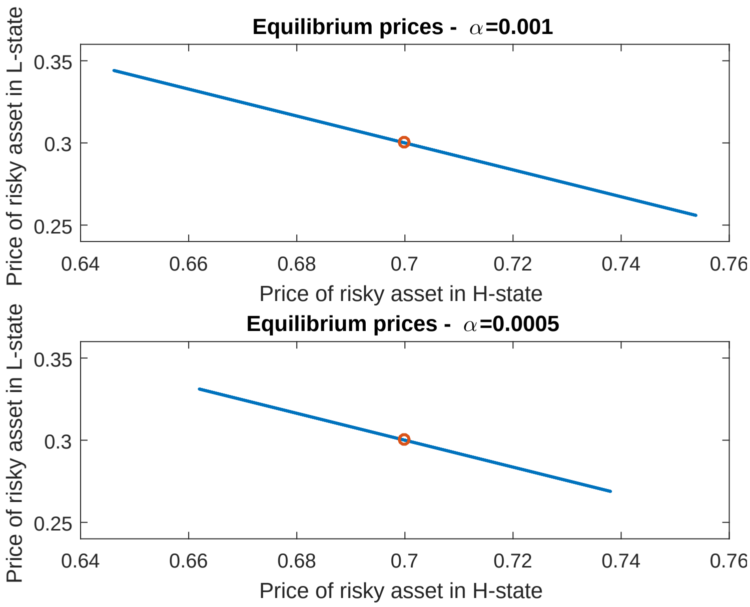
A.2
Prices without the random asset supply

If we drop out the assumption of an exogenous noisy asset supply, then there can be sunspost oscillations on equilibrium prices, which do not affect quantities. This is formalized in the proposition below.

Proposition: There are many equilibria with no information collection that satisfy the relationship:

$$-p(\theta_H)exp\left(-a\theta_H\bar{X}\right)[P_H^* - P_H] = p(\theta_L)exp\left(-a\theta_L\bar{X}\right)[P_L^* - P_L]$$

Figure A.1: Multiplicity of equilibria without the exogenous random asset supply



Under the κ setup, equilibrium prices are the whole line, characterized in the proposition.

Under the λ setup equilibrium prices are a segment of the line defined by the above equation. This can be illustrated in the numerical solution shown in figure A.2.

Market prices can fluctuate around the efficient market hypothesis in a such a way that is not profitable for the average trader to pay the informational cost of being aware of this fluctuation.

This shows that costly information processing can generate noisy behavior on prices, even without the random asset supply assumption.

Now we prove the proposition above:

Because there is no information collection, the following first order condition with respect to Z must hold:

$$\sum_j q_j \frac{\partial U(Z=0,Y_j)}{\partial Z_i} = 0$$

Therefore we must have that:

$$Pr(\theta_H)\frac{\partial U(Z_i=0,\theta_H)}{\partial Z_i} = -Pr(\theta_L)\frac{\partial U(Z_i=0,\theta_L)}{\partial Z_i}$$

Which noticing that:

$$\frac{\partial U(Z_i,Y_j)}{\partial Z_i} = -exp\left(-a(Z_i(\theta - PR_f) + R_f\bar{M} + \theta\bar{X} - a(\bar{X}+Z_i)^2\sigma_\epsilon^2/2)\right) \\ * \left[-a(\theta - PR_f) + a^2(\bar{X}+Z_i)\sigma_\epsilon^2\right]$$

Implies that

$$-Pr(\theta_H)exp\left(-a\theta_H\bar{X}\right)\left[-a(\theta_H - P_HR_f) + a^2\bar{X}\sigma_\epsilon^2\right] = \\ Pr(\theta_L)exp\left(-a\theta_L\bar{X}\right)\left[-a(\theta_L - PR_f) + a^2\bar{X}\sigma_\epsilon^2\right]$$

Multiply both sides by $1/aR_f$

$$-Pr(\theta_H)exp\left(-a\theta_H\bar{X}\right)\left[\frac{(\theta_H - P_HR_f)}{R_f} - \frac{a\bar{X}\sigma_\epsilon^2}{R_f}\right] = \\ Pr(\theta_L)exp\left(a\theta_L\bar{X}\right)\left[\frac{(\theta_L - PR_f)}{R_f} - \frac{\bar{X}\sigma_\epsilon^2}{R_f}\right]$$

Therefore, noticing that:

$$P^* = \left(\theta - a\bar{X}\sigma_\epsilon^2\right)\frac{1}{R_f}$$

$$-Pr(\theta_H)exp\left(-a\theta_H\bar{X}\right)[P_H^* - P_H] = Pr(\theta_L)exp\left(-a\theta_L\bar{X}\right)[P_L^* - P_L]$$

■

A.3
Non-existence with a finite number of agents

Definition: A Walrasian General Equilibrium with a finite number of agents is defined as : i) a price function $P(\omega_A)$, where ω_A denotes the aggregate state, ii) for each agent $k \in K$ a solution f_k^* of the rational inattention problem, such that:

$$Pr(\sum_{k=1}^{K} Z_k = 0) = 1 \qquad \text{for every aggregate state } \omega_A$$

Proposition: Let Y be a vector of random variables as defined before. Let there be J agents, and Z_k denote the vector of net demands of agent k. If $\{Z_1, Z_2, ..., Z_J\}|Y$ are independently distributed (i.e. each agent independently processes information), agents have strictly positive Shannon capacity and the information constraint is binding, then there is no price vector and allocation satisfying the definition above.

Proof: If $(Z_i|Y)_i^J$ are non-degenerate random variables, then they can only satisfy $\sum_{i=1}^{J} Z_i|Y = 0$ if they are not independently distributed, contradicting the assumption in the proposition. If $(Z_i|Y)_i^J$ are degenerate random variables, then $Z_i|Y$ is a function of Y. For that to be the case, the actions must form a partition of Y. If the information constraint is binding and the information capacity is strictly positive, this kind of strategy is not optimal. If the strategy space has one point of support, then no information is processed, so the agent is not using any amount of capacity, and the information constraint is not binding. If it has two or more points of support then, all points of the support of the strategy space have positive density (or probability) conditional on any possible realization of Y (this is shown in (11)). Thus, $\{Z_1, Z_2, ..., Z_J\}|Y$ cannot satisfy $\sum_{i=1}^{J} Z_i|Y = 0$. ■

A.4
Existence with an infinite number of replicas

Proposition: Let the number of replicas r of J types of agents grow to infinity. A general equilibrium must satisfy:

$$lim_{r\to\infty}Pr(\sum_{i=1}^{rJ} Z_i = 0) = 1$$

We will show that if agents process information independently, under the usual conditions, ie. preferences are continuous, strictly convex and strongly monotone and, for each good k $\sum_{i=1}^{rJ} w_{ik} > 0$ where w_{ik} is the agent i endowment of good k, there exists a general equilibrium. We will limit ourselves to the case where number of states is finite and x lies on a finite number of points. This is the case we numerically solve for.

Proof: We will apply a Weak Law of Large Numbers: if $Z_1, Z_2, ...$ are 2-2 uncorrelated, with uniformly bounded variances (that is, $VarZ_i \leq c$ for some finite c and for all i), then $\sum_{i=1}^{rJ} Z_i$ will converge in probability to $\sum_{j=1}^{J} E(Z_j|Y)$ which is analogous to the aggregate excess demand function. Thus, the general equilibrium will be represented by a system of correspondences such that $0 \in \mathbf{Z}(P) = \sum_{j=1}^{J} E(Z_j|Y)$ where $\mathbf{Z}(P) = (\mathbf{z_1}(P), \mathbf{z_2}(P), ..., \mathbf{z_k}(P))$, whose solution is guaranteed to exist under the following conditions:

i) **Z**(P) is upper-hemi continuous;

ii) **Z**(P) is homogeneous of degree zero;

iii) P***Z**(P)=0 for all P;

iv) there is an s>0 such that $\mathbf{z}_l(P) > -s$ for every commodity l and all P;

v) if $P^n \to P$, where $P \neq 0$, $p_l = 0$ for some l, then

$$max\{\mathbf{z}_1(P^n), ..., \mathbf{z}_L(P^n)\} \to \infty$$

We will show that the variance is uniformly bounded and those five conditions hold. We assume that given Y, $Z_1, Z_2, ...$ are independent random variables.

Lemma 1: $Var(Z_i|Y)$ is uniformly bounded.

Proof: Each vector of possible consumption bundles belongs to the budget set, which is compact and convex. Therefore, the vector of net demanded goods, given Y (where Y includes the realization of endowments and prices), belongs to a compact and convex set. Therefore, $supp(Z_i|Y)$ is a compact set and $Var(Z_i|Y)$ is finite. Take $c = \underset{i}{max} Var(Z_i|Y)$ which exists because the number of types J is finite. Thus $Var(Z_i|Y) \leq c$. ■

Lemma 2: $E[Z_i|Y]$ is a upper-hemi continuous correspondence of Y

Proof: Agent problem can be written as:

$$\underset{\{p_{ij}\}}{max} \sum_{ij} p_{ij} U(Z_i, Y_j)$$

$$subject\ to\ \sum_{ij} p_{ij} = q_j$$

$$I(Z;Y) = k$$

This is a jointly continuous objective function subject to a compact constraint. Therefore, from Berge Maximum theorem, we know that $\{p_{ij}\}^*$ is upper hemicontinuous. This in turn implies that $E(Z|Y)$ is upper hemicontinuous in $P(\omega)$.

■

Lemma 3: **Z**(P) is homogeneous of degree zero (on P).

Proof: As in the usual perfect information decision problem the relevant variable for the consumer is the normalized price vector. Thus agents don't make any difference between multiples of P. ■

Lemma 4: P***Z**(P)=0 for all P

Proof: For every i and every realization of (Z_i, Y) the budget constraint holds. Thus it holds for the conditional expectation of Z_i, $E(Z_i|Y)$. Summing over all agents we get the result above. ■

Lemma 5: There is an s>0 such that $\mathbf{z}_l(p) > -s$ for every commodity l and all p

Proof: Let w_{li} be the initial endowment of the good l by agent i, and c_{li} her consumption. Then assuming non negativity of consumption, we have: $\mathbf{z}_l(P) = \sum_{i=i}^{J} \mathbf{z}_{li}(P) = \sum_{i=i}^{J}(E(c_{li} + w_{li}|Y) \geq -\sum_{i=1}^{J} w_{li}$. Choose an s such that $s > \max_l \sum_{i=1}^{J} w_{li}$. ■

Lemma 6: if $P^n \to P$, where $P \neq 0$, $p_l = 0$ for some l, then

$$max\{\mathbf{z}_1(P^n), ..., \mathbf{z}_L(P^n)\} \to \infty$$

Proof: If for a given Y_1, P^n is such that there is p_l close to 0, then there will be at least one type of agent whose wealth is on average strictly positive, and provided his information capacity is greater than zero it will be worth for him to use his information capacity to know whether p_l is close to 0 and get an amount of utility that converges to ∞ when P^n goes to P. ■

By Proposition 17.C.1 ((29)) there is a function $P(\omega)$ for which $0 \in \mathbf{Z}(P)$. Thus, there exist a general equilibrium with rational inattentive agents for our last definition.■

We can extend the proposition above to show that if agents have arbitrary continuous beliefs over P, provided those beliefs cover the realized prices, then there is always a Walrasian general equilibrium as we defined it.

Lemmas 1 and 3-6 remain unaltered. We only need in the place of Lemma 2, the following proposition.

Lemma 2': $E[Z_i|Y]$ is a continuous function of Y, provided g(Y) is a continuous function.

Proof:

For the linear quadratic case, the proof is straightforward: If f(z|y) is a continuous functions of y, then $E(Z|Y) = \int zf(z|y)dz$ is a continuous function. Thus in the linear quadratic case, where f(z,y) is multivariate normal, E(Z|Y) is continuous.

Under broader conditions, if g(y) is continuous, then we use the following proposition ((11)):

If the information constraint is binding and if g(y)>0, then for a realized y, z is drawn from a conditional distribution with the following pdf:

$$f(z|y) = \frac{e^{U(z,y)/\lambda}}{E_{\hat{z}}[e^{U(\hat{z},y)/\lambda}]} \frac{f(z)}{g(y)}$$

Where λ is the Lagrange multiplier on the information constraint.

We had shown above that, for arbitrary beliefs about the distribution of prices, (provided g(y) is a continuous function and covers the realized price) there is a vector of prices that clears all the markets.

A.5 Non-binding information constraint

In a full information equilibrium, the demands are given by:

$$Z = \frac{(\theta - R_f P)}{a\sigma_\epsilon^2} - \bar{X}$$

Prices are given by:

$$P = \left(\theta - a\bar{\bar{X}}\sigma_\epsilon^2\right)\frac{1}{R_f}$$

Therefore, in a full information equilibrium the conditional demands of each agent are equal to the noisy asset supply at each state, that is $Z = \tilde{X}$ for each state. Conditional on the aggregate state, every agent demands the exactly quantity that matches the noise demands. This may sound trivial, but in fact this is not true for each agent when they are information constrained: they process information imperfectly, and are not sure which state they are in. Therefore, some agents demand more than the noise supply and some agents demands less.

The entropy in bits of the distribution of exogenous variables is

$$H(Y) = \sum_{\omega_A \in A} Pr(\omega_A) log_2(1/Pr(\omega_A)) = \sum_{1}^{4} 1/4 log_2(4) = 2 \text{ bits}$$

The mutual information I(Z;Y) of the optimal response in equilibrium is equal to:

$$I(Z;Y) = H(Y) - H(Y|Z) = 1 \text{ bit}$$

Since the optimal information strategy uses less information than the amount of exogenous uncertainty as measured by entropy, there is no need to know perfectly what is the aggregate state of the world, or what are the exactly values of every variable in Y. In particular, one can economize information by not processing all the information contained in prices.

The optimal signal can be the random asset supply itself. However, the optimal signal does not need to be formed with direct reference to the random asset supply. It can be based on prices, by adopting a signal S^* and actions Z of the form:

$$S^* = \begin{cases} 1 & \text{if } P = \left(\theta_H - a(\tilde{X}_H - \bar{X})\sigma_\epsilon^2\right) \frac{1}{R_f} \text{ or } \left(\theta_L - a(\tilde{X}_H - \bar{X})\sigma_\epsilon^2\right) \frac{1}{R_f} \\ 0 & \text{otherwise} \end{cases}$$

$$Z^* = \begin{cases} \tilde{X}_H & \text{if } S^* = 1 \\ \tilde{X}_L & \text{if } S^* = 0 \end{cases}$$

or more generally, the optimal signal is any random variable S such that in states $(\theta_H, \tilde{X}_H)$ and $(\theta_L, \tilde{X}_H)$ is equal to s_1 and in states $(\theta_H, \tilde{X}_L)$ and $(\theta_L, \tilde{X}_L)$ is equal to s_2, where $s_1 \neq s_2$.

In equilibrium there is no need to know the value of the endowment return. This feature however is knife-edged, hinging on the hypothesis of an exogenous asset supply that is completely inelastic to prices. With some elasticity, which can be introduced, for instance, if we postulate an exogenous risk-free asset supply instead of an exogenous risky asset supply, agents do process in equilibrium information about the endowment return.

A.6 Moments of excess returns

Some moments are straightforward to calculate given the parameters and the solution for equilibrium prices. Others are not so obvious, so we give a complete description of them here.

Mean:

$$\mu = E\left[\frac{\theta+\epsilon}{P} - rf\right] = E\left[\frac{\theta}{P} - rf\right]$$

Let us define:

$$\hat{\mu} = \mu + rf$$

Variance:

$$\sigma^2 = V\left[\frac{\theta+\epsilon}{P}\right] = E\left[\left(\frac{\theta+\epsilon}{P}\right)^2\right] - \tilde{\mu}^2$$

$$E\left[\left(\frac{\theta+\epsilon}{P}\right)^2\right] = \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\}) \int \left(\frac{\theta+\epsilon}{P}\right)^2 f(\epsilon)d\epsilon = \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\}) \left(\frac{\theta^2+\sigma_\epsilon^2}{P^2}\right)$$

Skewness:

$$E\left[\frac{\frac{\theta+\epsilon}{P} - rf - \mu}{\sigma}\right]^3 = \sigma^{-3} \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\}) \int \left(\frac{\theta+\epsilon}{P} - \tilde{\mu}\right)^3 f(\epsilon)d\epsilon$$

$$= \sigma^{-3} \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\})$$

$$\int \left(\frac{\theta^3}{P^3} + \frac{3\epsilon\theta^2}{P^3} + \frac{3\epsilon^2\theta}{P^3} + \frac{\epsilon^3}{P^3} - \frac{3\theta^2\tilde{\mu}}{P^2} - \frac{6\epsilon\theta\tilde{\mu}}{P^2} - \frac{3\epsilon^2\tilde{\mu}}{P^2} + \frac{3\theta\tilde{\mu}^2}{P} + \frac{3\epsilon\tilde{\mu}^2}{P} - \tilde{\mu}^3\right) f\epsilon d\epsilon$$

$$= \sigma^{-3} \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\}) \left[\frac{\theta^3}{P^3} + \frac{3\sigma_\epsilon^2\theta}{P^3} - \frac{3\theta^2\tilde{\mu}}{P^2} - \frac{3\sigma_\epsilon^2\tilde{\mu}}{P^2} + \frac{3\theta\tilde{\mu}^2}{P} - \tilde{\mu}^3\right]$$

Kurtosis:

$$E\left[\frac{\frac{\theta+\epsilon}{P} - rf - \mu}{\sigma}\right]^4 = E\left[\frac{\frac{\theta+\epsilon}{P} - \tilde{\mu}}{\sigma}\right]^4 = \sigma^{-4} \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\}) \int \left(\frac{\theta+\epsilon}{P} - \tilde{\mu}\right)^4 f\epsilon d\epsilon$$

$$= \sigma^{-4} \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\})$$

$$\int \left(\frac{\theta^4}{P^4} + \frac{4\epsilon\theta^3}{P^4} + \frac{6\epsilon^2\theta^2}{P^4} + \frac{4\epsilon^3\theta}{P^4} + \frac{\epsilon^4}{P^4} - \frac{4\theta^3\tilde{\mu}}{P^3} - \frac{12\epsilon\theta^2\tilde{\mu}}{P^3} \right.$$

$$\left. - \frac{12\epsilon^2\theta\tilde{\mu}}{P^3} - \frac{4\epsilon^3\tilde{\mu}}{P^3} + \frac{6\theta^2\tilde{\mu}^2}{P^2} + \frac{12\epsilon\theta\tilde{\mu}^2}{P^2} + \frac{6\epsilon^2\tilde{\mu}^2}{P^2} - \frac{4\theta\tilde{\mu}^3}{P} - \frac{4\epsilon\tilde{\mu}^3}{P} + \tilde{\mu}^4 \right) f\epsilon d\epsilon$$

$$= \sigma^{-4} \sum_{\{\theta,\tilde{X}\}} Pr(\{\theta,\tilde{X}\})$$

$$\left(\frac{\theta^4}{P^4} + \frac{6\sigma_\epsilon^2\theta^2}{P^4} + \frac{3\sigma_\epsilon^4}{P^4} - \frac{4\theta^3\tilde{\mu}}{P^3} - \frac{12\sigma_\epsilon^2\theta\tilde{\mu}}{P^3} + \frac{6\theta^2\tilde{\mu}^2}{P^2} + \frac{6\sigma_\epsilon^2\tilde{\mu}^2}{P^2} - \frac{4\theta\tilde{\mu}^3}{P} + \tilde{\mu}^4 \right)$$

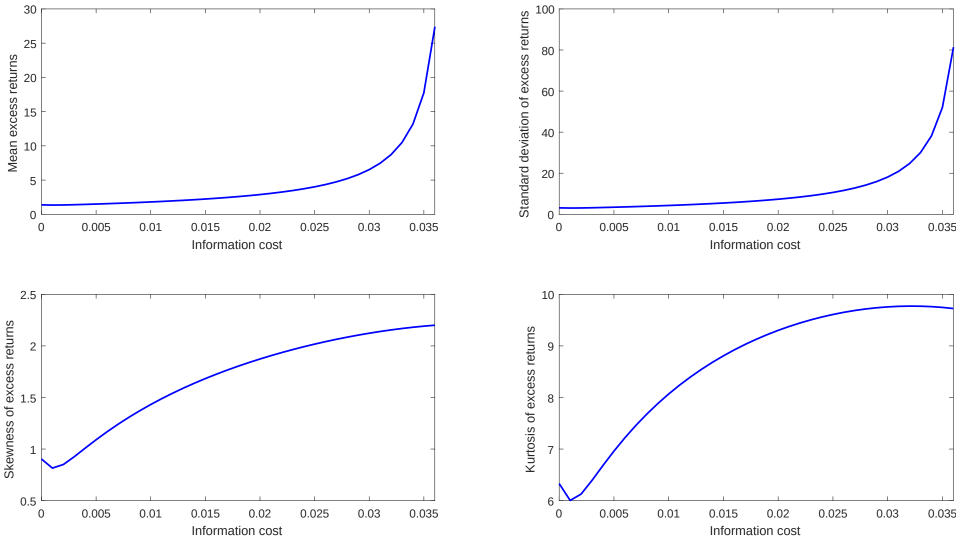


Figure A.2: Moments of the distribution of excess returns

A.7
Risk-free asset random supply

Instead of assuming that the risky asset supply is random, here we assume that is the risk-free asset supply that is random (M_{noise}).

In this version the supply of the risk free asset is exogenously random, so the noise supply of the risky asset is not inelastic to prices. When the noise buys the risky asset, it pays with the risk-free assets, so there is a hidden budget constraint for the noise. This budget constraint implies that $\tilde{X} * P = -M_{noise}$ Therefore, $\tilde{X} = -M_{noise}/P$. This is the simplest way of introducing some elasticity to prices to the risky asset supply, in this Grossman and Stiglitz like model. The main change this elasticity introduces is that agents again now pay attention to endowment return in equilibrium even when the information constraint is binding, provided they have a sufficiently high information capacity.

The figures below illustrate numerical solutions. A expression for full information equilibrium prices is given in the last section of the Appendix.

A.7.1
Varying the information capacity

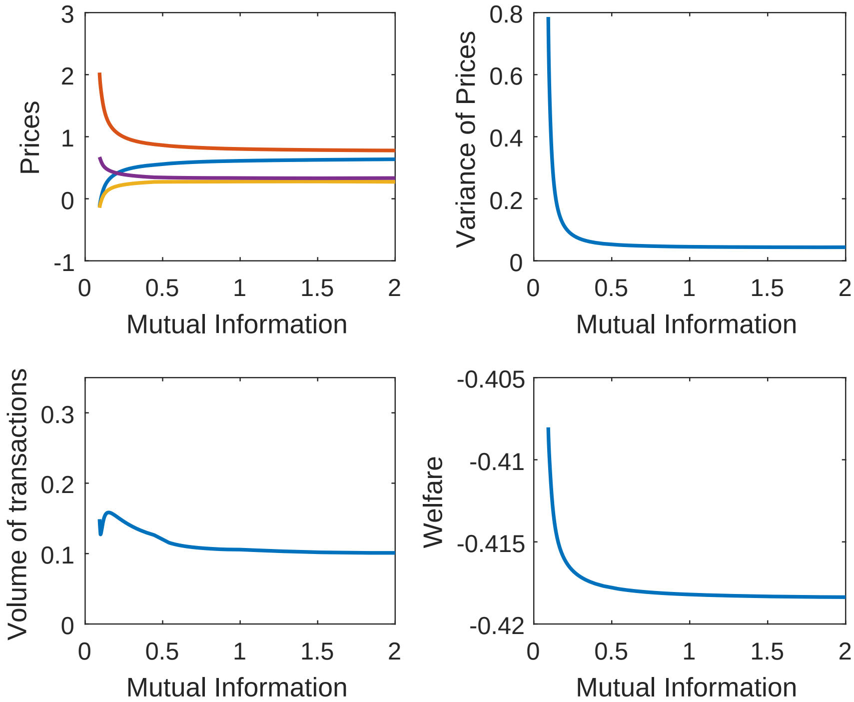


Figure A.3: Prices and volume as a function of the information capacity of agents

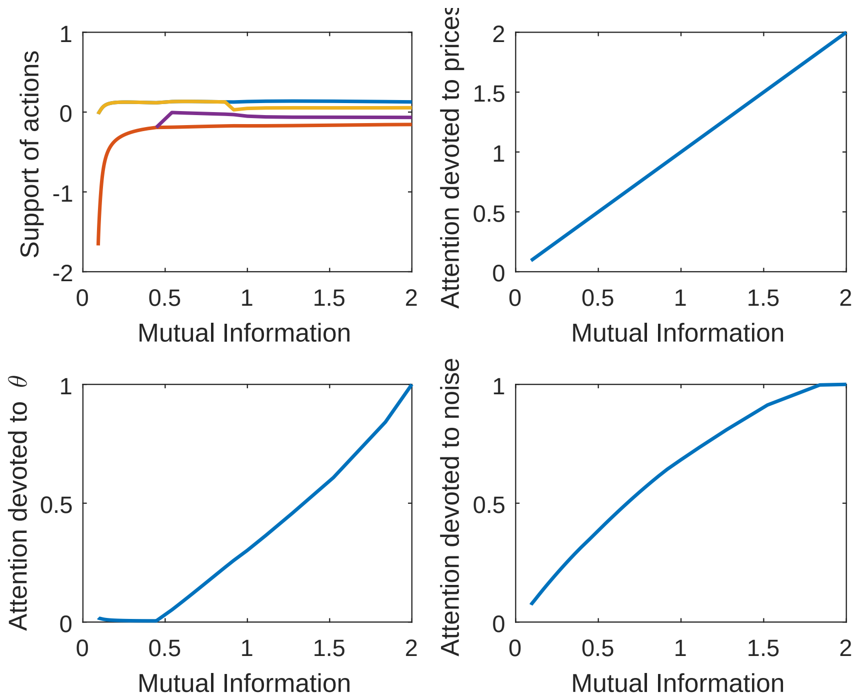


Figure A.4: Actions and what agents pay attention to

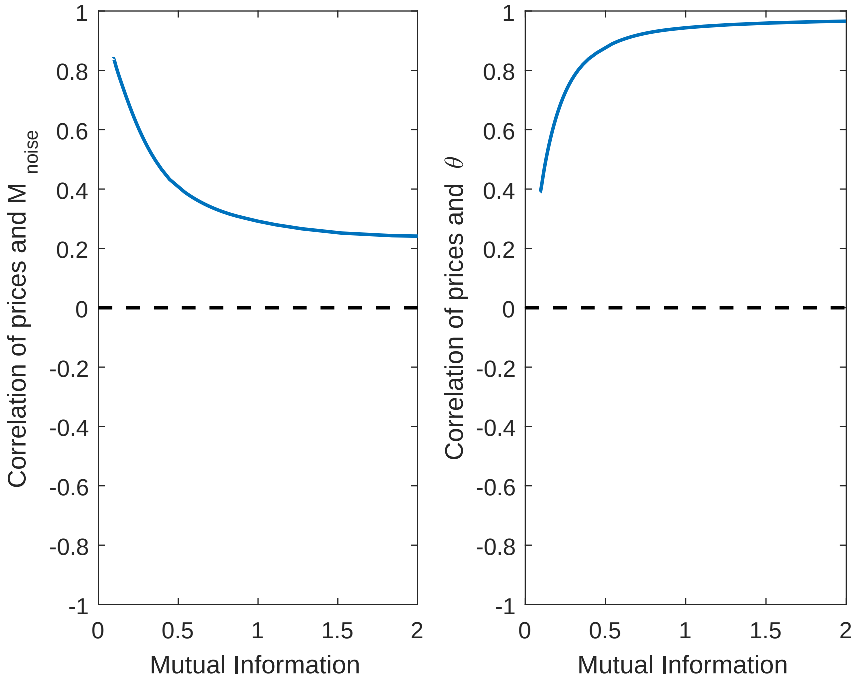


Figure A.5: Correlation of prices, θ and M_{noise}

A.7.2 Varying the information cost

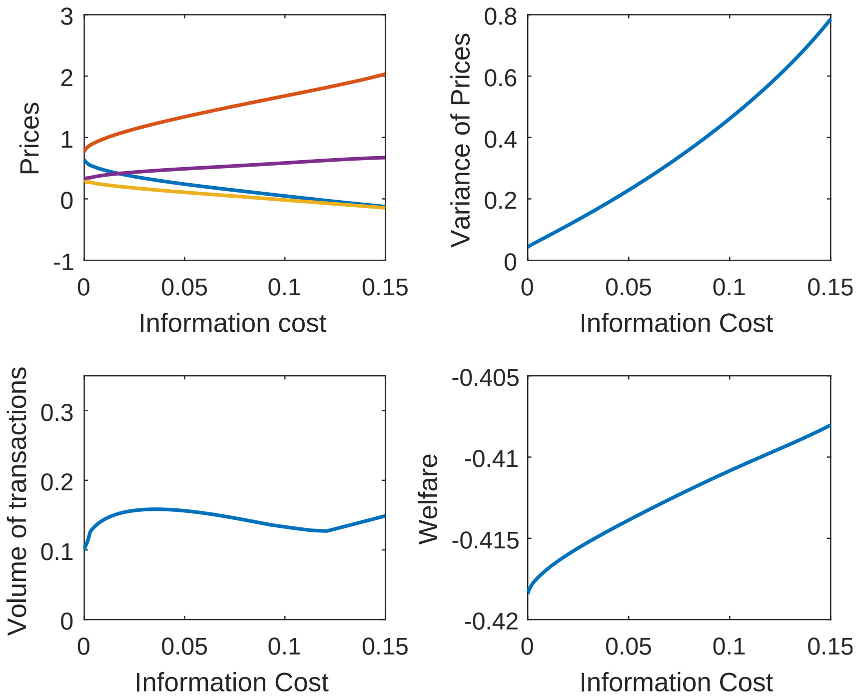


Figure A.6: Prices and volume as a function of the information cost of agents

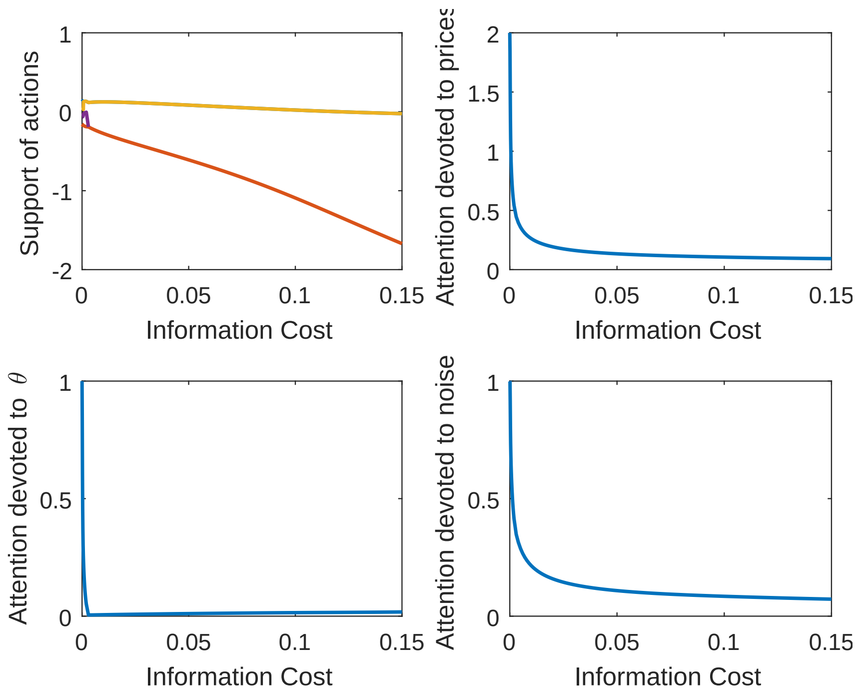


Figure A.7: Actions and what agents pay attention to

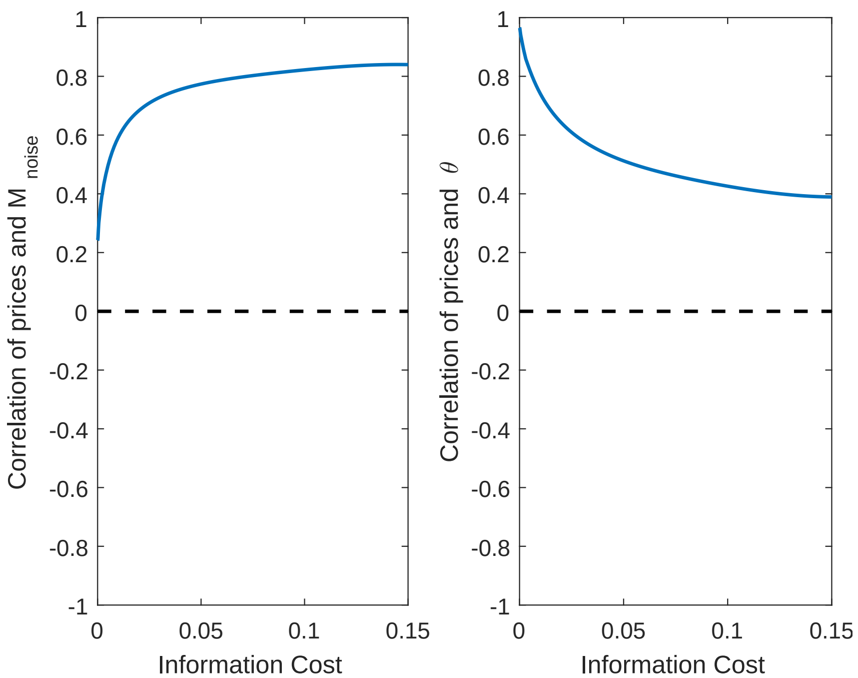


Figure A.8: Correlation of prices, θ and M_{noise}

A.7.3 Full information prices under the random risk-free asset supply setup

We have showed that the following expression holds for full information prices:

$$P = \left(\theta - a\bar{\bar{X}}\sigma_\epsilon^2\right)\frac{1}{R_f}$$

Where $\bar{\bar{X}}$ denotes the aggregate amount of the risky asset in the economy.

Noticing that, because noise transactions are paid with the risky-free asset, the following holds:

$$-P * \tilde{X} = M_{noise}$$

Therefore, we can write:

$$P = \left(\theta - a(\bar{X} - M_{noise}/P)\sigma_{\epsilon}^2\right)\frac{1}{R_f}$$

Which implies in the second-order equation:

$$P^2 - \left(\theta - a(\bar{X})\sigma_{\epsilon}^2\right)\frac{P}{R_f} - \frac{a\sigma_{\epsilon}^2 M_{noise}}{R_f} = 0$$

Whose solution is:

$$P = \frac{\left(\theta - a(\bar{X})\sigma_{\epsilon}^2\right)\frac{1}{R_f} \pm \sqrt{\left[\left(\theta - a(\bar{X})\sigma_{\epsilon}^2\right)\frac{1}{R_f}\right]^2 + 4\frac{a\sigma_{\epsilon}^2 M_{noise}}{R_f}}}{2}$$

B
Computational Appendix

The archive appendix.rar contains the routines.