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Equilibrium with default-dependent
credit constraints

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EQUILIBRIUM WITH DEFAULT-DEPENDENT CREDIT CONSTRAINTS

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ABSTRACT. We state an infinite horizon sequential markets model with real assets in positive net supply and subject to credit risk. By introducing default-dependent borrowing constraints, we show the existence of equilibrium.

1. INTRODUCTION

During the last years, the study of sequential markets economies with infinite horizon has become one of the fields of interest in general equilibrium theory, leading to an increasingly literature on this issue. By imposing conditions that prevent debt accumulation from raising through the time, several authors have proved the existence of equilibrium in infinite horizon economies with financial markets. However, with the aim of avoiding *Ponzi Schemes*, both transversality conditions and debt constraints have been used (see, for instance, Magill and Quinzii (1994, 1996), Levine and Zame (1996) and Hernandez and Santos (1996)). These constraints, which may be either exogenous or made endogenous as a function of the prices, are stated just as agents' budget constraints but they are not inferred as a consequence from a supposed underlying rationality. Nevertheless, when penalties for default are supposed to be infinite, agents honor all their commitments and, therefore, there is no natural reason to limit the amount of credit.

Hence, it is of interest to present and analyze models where the required constraints are compatible either with the rationality of the agents or with the usual default-control market mechanisms. Essentially, when default is allowed, any condition that bounds the progressive increase of the debt, although it is not deduced from individuals rationality, may be understood as a natural market constraint provided that debtors are free to pay their total debts. Following this approach and without *a priori* imposing debt constraints or transversality conditions, Araujo, Páscoa and Torres-Martínez (2002) obtained existence of equilibria in models where agents may not pay their obligations, but markets prescribe guarantees, by means of collateral requirements, at the sale of

every asset. Recently, Páscoa and Seghir (2007) show that, when agents suffer extra-economic penalties in case of default, equilibria do still exist provided that penalties are not too harsh. Further, Braido (2007) proves that in infinite horizon stationary economies equilibrium exists when default-dependent short-sales constraints are imposed, asset by asset.

In this paper, we address a general equilibrium model with assets in positive net supply subject to credit risk. Borrowers may pay just a percentage of their debts but then they suffer *borrowing constraints* which depend on their amount of default. More precisely, at each period and state of nature, the amount of borrowing is bounded in accordance with three factors: the aggregated amount of default, the original promises made by the borrower and the amount of resources that was really payed. For simplicity, we assume that each type of debt contract is pooled into only one security, that is bought by lenders and makes endogenous payments. Moreover, we show the existence of equilibria in which derivatives are negotiated by the same unitary price as the underlying loans and, therefore, the quantity of debt contracts sold by borrowers equates the total number of derivatives bought by lenders. Our framework is general enough to allow for non-stationary economies, hyperbolic inter-temporal discounting and heterogeneous beliefs about uncertainty.

The rest of the paper is organized as follows: Section 2 describes the model. Section 3 states the main result of existence of equilibrium and discusses its implications in the literature. A final Appendix includes the proof of our main result.

2. The Economy

We consider a discrete time economy with infinite horizon. Let $S \neq \emptyset$ be the set of states of nature. At each date $t \in \{0, 1, \ldots\}$ individuals have common information about the realization of the uncertainty, which is given by a finite partition $\mathcal{F}_t$ of S. We assume that, for each $t \geq 0$, $\mathcal{F}_{t+1}$ is finer than $\mathcal{F}_t$, with $\mathcal{F}_0 = S$.

A pair $\xi = (t, \sigma)$, where $t \geq 0$ and $\sigma \in \mathcal{F}_t$, is called a *node* of the economy. Denote by $D = \{(t, \sigma) : t \geq 0, \sigma \in \mathcal{F}_t\}$ the set of all nodes, called the *event-tree*. Given $\xi = (t, \sigma)$ and $\mu = (t', \sigma')$, we say that μ is a *successor* of ξ, and we write $\mu \geq \xi$, if $t' \geq t$ and $\sigma' \subset \sigma$. Let $t(\xi)$ be the date associated to ξ and ξ^+ be the set of immediate successors of ξ, that is, the set of nodes $\mu \geq \xi$, where $t(\mu) = t(\xi) + 1$. The (unique) predecessor of ξ is denoted by ξ^- and ξ_0 is the node at $t = 0$. Let $D(\xi) := \{\mu \in D : \mu \geq \xi\}$, $D^T(\xi) := \{\mu \in D(\xi) : t(\mu) \leq T + t(\xi)\}$ and $D_T(\xi) := \{\mu \in D(\xi) : t(\mu) = T + t(\xi)\}$.

At each $\xi \in D$ there is a finite ordered set, L, of perishable commodities that can be traded in spot markets. Let $p(\xi) = (p_l(\xi); l \in L) \in \mathbb{R}_+^L$ be the price system of goods at ξ. Also, the process of commodity prices is denoted by $p = (p(\xi); \xi \in D)$.

We denote by J the set of debt contracts in the economy, which is composed only by short-lived real assets. Further, at each $\xi \in D$, there is a finite ordered set $J(\xi) \subset J$ of assets available for

borrowers. Each $j \in J(\xi)$ is subject to default and characterized by the (unitary) real promises, $A(\mu, j) \in \mathbb{R}_+^L$, where $\mu \in \xi^+$. We assume that, for each $j \in J(\xi)$, $(A(\mu, j), \mu \in \xi^+) \neq 0$.

Although in financial markets debt promises are usually securitized by families of derivatives, for simplicity, we will assume that each debt contract is pooled into only one derivative that is negotiated by the same unitary price. Thus, for notational convenience, we identify debt contracts with securities negotiated by lenders. Let $q(\xi) = (q_j(\xi); j \in J(\xi))$ be the vector of asset prices at ξ. Also, $q := (q(\xi); \xi \in D)$. Define $D(J) = \{(\xi, j) \in D \times J : j \in J(\xi)\}$ and $D_-(J) = \{(\xi, j) \in D \times J : j \in J(\xi^-)\}$.

There are a finite number of agents, $h \in H$, that can trade securities and commodities at each node in the event-tree. Each $h \in H$ is characterized by her physical and financial endowments, $(w^h(\xi), e^h(\xi)) \in \mathbb{R}_{++}^L \times \mathbb{R}_+^{J(\xi)}$, at each node $\xi \in D$, and by her preferences on consumption, which are represented by an utility function $U^h : \mathbb{R}_+^{D \times L} \to \mathbb{R}_+ \cup \{+\infty\}$. For convenience of notation, we will denote by $W(\xi) := \sum_{h \in H}(w^h(\xi) + \sum_{j \in J(\xi^-)} A(\xi, j)e_j^h(\xi))$ the aggregated physical endowments that are in the economy, at node $\xi > \xi_0$, in the absence of default. At ξ_0 aggregated endowments are given by $W(\xi_0) = \sum_{h \in H} w^h(\xi_0)$.

Each borrower decides the amount of his payments whereas the lenders expect to receive endogenous returns, that take into account the mean rate of default. Agents face exogenous constraints on the amount of credit. These constraints are given in real terms and depend on the aggregate amount of default, on the original promises made by the borrower and on the effective payments made.

More formally, consider that an agent $h \in H$ sold a vector of securities $\varphi^h(\xi) := (\varphi^h(\xi, j); j \in J(\xi))$ at each node ξ. Then, she promised to pay $T_\mu(\varphi^h(\xi)) := \sum_{j \in J(\xi)} p(\mu)A(\mu, j)\varphi_j^h(\xi)$ at node $\mu \in \xi^+$. However, she can decide to delivers only an amount $t_\mu(\tilde{\varphi}^h(\mu)) := \sum_{j \in J(\xi)} p(\mu)A(\mu, j)\tilde{\varphi}_j^h(\mu)$, where $\tilde{\varphi}^h(\mu) \leq \varphi^h(\xi)$[1]. Thus, with the aim of prevent lenders for excess of losses, we assume that agent h perfect foresight the whole amount of default at $\mu > \xi_0$, namely m_μ, and faces the following borrowing constraints along the event-tree,

$$
\begin{aligned}
q(\xi_0)\varphi^h(\xi_0) &\leq p(\xi_0)M, \\
q(\mu)\varphi^h(\mu) &\leq \mathbb{C}_\mu^h\left[T_\mu(\varphi^h(\mu^-)), \; t_\mu(\tilde{\varphi}^h(\mu)), \; m_\mu\right]p(\mu)M, \quad \forall \mu \in D \setminus \{\xi_0\},
\end{aligned}
$$

where $M \in \mathbb{R}_{++}^L$ and the functions $\mathbb{C}_\mu^h : \mathbb{A} \subset \mathbb{R}_+^3 \to \mathbb{R}_{++}$ are exogenously fixed with $\mathbb{A} := \{(T, t, m) \in \mathbb{R}_+^3; 0 \leq T - t \leq m\}$.

On the other hand, each lender of asset $j \in J(\xi)$ will expect to receive, at each $\mu \in \xi^+$, a percentage $\alpha_j(\mu)$ of the original promises. Individuals take as given the anonymous payment rates $\alpha := (\alpha_j(\xi), (\xi, j) \in D(J)) \in [0, 1]_+^{D(J)}$, which are determined at equilibrium. Further, agents perfect foresight the mean percentage of the original promises that are honored, given the (endogenous)

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[1] Note that, in this case, the individual default at $\mu \in \xi^+$ is given by $T_\mu(\varphi^h(\xi)) - t_\mu(\tilde{\varphi}^h(\mu))$.

deliveries made by the borrowers and the returns received by financial endowments. For simplicity, we assume that financial endowments suffer default following anonymous payment rates.

Finally, to avoid over-pessimistic beliefs about the rates of default, we also assume that markets can always seize resources in order to assure that borrowers will pay at least an $\underline{\alpha}_{\mu,j} \in (0,1)$ percentage of their financial commitments on asset $j \in J(\xi)$ at any node $\mu \in \xi^{+}$.[2]

Let $\mathbb{E} := \mathbb{R}_{+}^{D \times L} \times \mathbb{R}_{+}^{D(J)} \times \mathbb{R}_{+}^{D(J)} \times \mathbb{R}_{+}^{D_{-}(J)}$ be the space of individual's allocations. It follows from description above that, given prices (p,q), rates of payment α and aggregated amounts of default $m := (m_{\xi}; \xi > \xi_0)$, each $h \in H$ maximizes her preferences by choosing an allocation $(x^h, \theta^h, \varphi^h, \tilde{\varphi}^h) := \left((x^h(\xi), \theta^h(\xi), \varphi^h(\xi), \tilde{\varphi}^h(\xi)); \xi \in D \right)$ in her budget set $B^h(p,q,\alpha,m)$, defined as the set of plans $(x, \theta, \varphi, \tilde{\varphi}) \in \mathbb{E}$ such that, for each $\xi \in D$,

$$p(\xi)\left(x(\xi) - w^h(\xi)\right) + q(\xi)\left(\theta(\xi) - \varphi(\xi) - e^h(\xi)\right) \leq \sum_{j \in J(\xi^{-})} \left(\alpha_j(\xi)p(\xi)A(\xi,j)\theta_j(\xi^{-}) - p(\xi)A(\xi,j)\tilde{\varphi}_j(\xi)\right);$$

$$\tilde{\varphi}_j(\mu) \in [\underline{\alpha}_{\mu,j}\,\varphi_j(\xi), \varphi_j(\xi)], \quad \forall \mu \in \xi^{+}, \forall j \in J(\xi);$$

$$q(\xi_0)\varphi(\xi_0) \leq p(\xi_0)M;$$

$$q(\mu)\varphi(\mu) \leq \mathbb{C}_{\mu}^h\left[T_{\mu}(\varphi(\xi)),\ t_{\mu}(\tilde{\varphi}(\mu)),\ m_{\mu}\right]p(\mu)M, \quad \forall \mu \in \xi^{+};$$

where $(\theta(\xi_0^{-}), \varphi(\xi_0^{-})) = 0$. Moreover, $x^h(\xi) = (x_l^h(\xi); l \in L)$ is the consumption bundle of agent h at ξ. Analogously, $\theta_j^h(\xi)$ denotes the quantity of asset $j \in J(\xi)$ that agent h buys at ξ.

DEFINITION. An *equilibrium* for our economy is given by a vector of prices (p,q), a vector of anonymous payment rates and amounts of default, (α, m), jointly with allocations $\left((x^h, \theta^h, \varphi^h, \tilde{\varphi}^h); h \in H\right)$, such that:

(a) For each agent $h \in H$, $(x^h, \theta^h, \varphi^h, \tilde{\varphi}^h) \in \mathrm{argmax}_{(x,\theta,\varphi,\tilde{\varphi}) \in B^h(p,q,\alpha,m)}\ U^h(x)$.

(b) At each $\xi \in D$, physical and asset markets clear,

$$\sum_{h \in H} x^h(\xi_0) = \sum_{h \in H} w^h(\xi_0);$$

$$\sum_{h \in H} x^h(\xi) = \sum_{h \in H} w^h(\xi) + \sum_{j \in J(\xi^{-})} \alpha_j(\xi)\,A(\xi,j)\sum_{h \in H} e_j^h(\xi^{-}), \quad \forall \xi > \xi_0;$$

$$\sum_{h \in H} \theta^h(\xi) = \sum_{h \in H} e^h(\xi) + \sum_{h \in H} \varphi^h(\xi).$$

(c) Each individual perfect foresight anonymous rates of payment, that is,

$$\alpha_j(\xi)\sum_{h \in H} \theta_j^h(\xi^{-}) = \alpha_j(\xi)\sum_{h \in H} e_j^h(\xi^{-}) + \sum_{h \in H} \tilde{\varphi}_j^h(\xi), \quad \forall(\xi,j) \in D(J).$$

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[2]Micro-foundations to this upper bound on the rates of default can be obtained by introducing durable goods and physical guarantees, in the form of collateral requirements, as in Araujo, Páscoa and Torres-Martínez (2002), or by assuming that agents suffer extra-economic penalties which are proportional to the amount of default, as in Páscoa and Seghir (2007). For simplicity, we will maintain the parameters $\underline{\alpha}_{\mu,j}$ as exogenously given.

(d) Agents perfect foresight the total amount of default, i.e., for each $\xi \in D$,

$$m_\mu = \sum_{h \in H} \left(T_\mu(\varphi^h(\xi)) - t_\mu(\tilde{\varphi}^h(\mu)) \right), \quad \forall \mu \in \xi^+.$$

3. EXISTENCE OF EQUILIBRIUM

Our main result is obtain existence of equilibria for our economy.

THEOREM. There is an equilibrium in our economy provided that the following assumptions hold,

A. There exists $\underline{w} \in \mathbb{R}_{++}^L$ such that, for each $(\xi, h) \in D \times H$, $w^h(\xi) \geq \underline{w}$. Moreover, assets have positive net supply, i.e. $\sum_{h \in H} e_j^h(\xi) > 0$, for every $(\xi, j) \in D(J)$.

B. For each $h \in H$, $U^h(x) = \sum_{\xi \in D} u^h(\xi, x(\xi))$, where for every node $\xi \in D$ the function $u^h(\xi, \cdot) : \mathbb{R}_+^L \to \mathbb{R}_+$ is continuous, concave, strictly increasing and

$$\lim_{x \in \mathbb{R}_{++}^L; \|x\|_\infty \to +\infty} u^h(\xi, x) = +\infty.$$

Moreover, $\sum_{\xi \in D} u^h(\xi, W(\xi)) < +\infty$.

C. Functions $(\mathbb{C}_\mu^h)_{\{h \in H, \mu > \xi_0\}}$ are continuous, concave in the first two coordinates and uniformly bounded from above. Further, at each node $\mu > \xi_0$, there exists $\tau(\mu) \in \mathbb{R}_+ \setminus \{0\}$ such that $\mathbb{C}_\mu^h(a, a, m) \geq \tau(\mu)$ for every pair $(a, m) \in \mathbb{R}_+^2$.

The hypotheses on physical endowments included in Assumption A are standard in the related literature. The positive net supply of assets is adequate to assure the non-emptiness of the interior of budget correspondences, as we will have an (uniform) upper bound in financial prices, when commodities prices belong to the simplex.[3]

Utilities are separable (Assumption B) in order to obtain an equilibrium as a limit of equilibria in a sequence of (finite horizon) truncated economies. Note that, as inter-temporal discounted factors do not need to be constant, we allow for hyperbolic discounting in our model.

Now, since utilities explodes when consumption increases, we will have, at equilibrium, a uniform lower bound for asset prices. In fact, if an asset $j \in J(\xi)$ has a price $q_j(\xi) < \underline{q}$, then each $h \in H$ can get the consumption $w^h(\mu) + \underline{\alpha}_{\mu,j} A(\mu, j) \frac{p(\xi) w^h(\xi)}{\underline{q}}$, at each node $\mu \in \xi^+$, by buying $\frac{p(\xi) w^h(\xi)}{\underline{q}}$ units of such a security at node ξ. Thus, if $\underline{q}$ is small enough, individuals can increase their utilities above the maximum level compatible with equilibrium, namely, $\max_{h \in H} \sum_{\xi \in D} u^h(\xi, W(\xi))$.

Assumption C requires minimal conditions on the credit constraint functions in order to maintain convexity properties of our model. Of course, when an agent does not make default on her promises (or does not make promises) at a node μ, it seems natural that a minimum percentage of the original (real) amount of credit remains available, independently of the aggregated amount of default.

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[3]If we extend our framework to allow for assets in zero net supply, we do still can assure the existence of an uniform upper bound for financial prices provided that, as in Braido (2007), the following requirement on preferences holds: $\lim_{\|x\|_{min} \to 0} u^h(\xi, x) = -\infty$, where $\|x\|_{min} = \min_{l \in L} x_l$, for any $x = (x_l; l \in L) \geq 0$,

However, even lenders may suffer future credit constraints when there are default in the economy, that is, we allow for $\mathbb{C}_\mu^h(0, 0, m) < 1$, for each $m > 0$.

Although it is not necessary for obtaining equilibrium existence, we can consider credit functions $\mathbb{C}_\mu^h$ that are non-increasing in the total amount of debt (first coordinate) and non-decreasing in the other two variables, namely, effective payments and total amount of default. In particular, any continuous, concave and bounded application $g : \mathbb{R}_+ \to \mathbb{R}_+$ defines a credit function $\mathbb{C}_\mu^h(T, t, m) = g(T - t)$ that depends only on the individual amount of default.

We also point out that our proof of equilibrium existence relies crucially on the upper bound on credit constraint functions. In fact, if the amount of borrowing may be expanded infinitely along the event-tree, even without default, agents can enter into Ponzi schemes by paying previous debts with new credit. Finally, note that if we attempt to extend our model to allow for credit constraints which depend on the history of default, then it becomes important to establish explicit repayment rules of past debts. Actually, there is no reason to restrict the access to credit market to those debtors who have resources enough to pay previous commitments.

APPENDIX. PROOF OF EQUILIBRIUM EXISTENCE.

We essentially follow the technic of proof of equilibrium existence used by Moreno-García and Torres-Martínez (2006) in models without default. Thus, we start by defining a finite horizon truncation of our initial economy.

Truncated economies. Given $T \in \mathbb{N}$, let $\mathcal{E}^T$ be a truncated economy, in which agents live in the (restricted) event-tree $D^T(\xi_0)$. The set of available assets at $\xi \in D^{T-1}(\xi_0)$ is given by $J^T(\xi) = J(\xi)$. At the terminal nodes, $\xi \in D_T(\xi_0)$, we consider $J^T(\xi) = \emptyset$. Let $D^T(J) = \{(\xi, j) \in D^T(\xi_0) \times J : j \in J^T(\xi)\}$ and $D_-^T(J) = \{(\mu, j) \in D^T(\xi_0) \times J : j \in J^T(\xi), \mu \in \xi^+\}$. We consider prices, rates of payment and amount of default, (p, q, α, m), belonging to

$$\mathbb{P}^T := \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^L \times \mathbb{R}_+^{J^T(\xi)} \right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^L \times \prod_{(\mu, j) \in D_-^T(J)} [\underline{\alpha}_{\mu,j}, 1] \times \left(\mathbb{R}_+^{D^T(\xi_0) \setminus \{\xi_0\}} \right),$$

where $\Delta_+^L := \{p \in \mathbb{R}_+^L : \|p\|_\Sigma = 1\}$. Given $(p, q, \alpha, m) \in \mathbb{P}^T$, the objective of agent h is to solve,

$$(P^{h,T}) \quad \begin{array}{l} \max \sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi)) \\ \text{s.t.} \quad \begin{cases} g_\xi^{h,T}\left(y(\xi), y(\xi^-); p, q, \alpha\right) & \leq & 0, & \forall \xi \in D^T(\xi_0), \\ \tilde{\varphi}_j(\mu) \in [\underline{\alpha}_{\mu,j}\, \varphi_j(\xi), \varphi_j(\xi)], & & & \forall (\xi, j) \in D^T(J), \forall \mu \in \xi^+; \\ q(\mu)\varphi(\mu) - \mathbb{C}_\mu^h\left[T_\mu(\varphi(\xi)),\, t_\mu(\tilde{\varphi}(\mu)),\, m_\mu\right] p(\mu)M & \leq & 0, & \forall \mu \in \xi^+, \forall \xi \in D^T(\xi_0); \\ y(\xi) = (x(\xi), \theta(\xi), \varphi(\xi), \tilde{\varphi}(\xi)) & \geq & 0, & \forall \xi \in D^T(\xi_0), \\ (\theta(\xi), \varphi(\xi)) & = & 0, & \forall \xi \in D_T(\xi_0), \end{cases} \end{array}$$

where the inequality $g_\xi^{h,T}\left(y(\xi), y(\xi^-); p, q, \alpha\right) \leq 0$ represents the budget constraint of agent h at node $\xi \in D^T(\xi_0)$. Now, let $B^{h,T}(p, q, \alpha, m)$ be the truncated budget set of agent h, i.e., the set of plans $(y(\xi))_{\xi \in D^T(\xi_0)}$ that satisfy the restrictions of problem $P^{h,T}$ above.

DEFINITION A1. An *equilibrium* for $\mathcal{E}^T$ is given by a vector $(p^T, q^T, \alpha^T, m^T) \in \mathbb{P}^T$ and individual allocations

$$y^{h,T} = (x^{h,T}(\xi), \theta^{h,T}(\xi), \varphi^{h,T}(\xi), \tilde{\varphi}^{h,T}(\xi))_{\xi \in D^T(\xi_0)} \in \mathbb{E}^T := \mathbb{R}_+^{D^T(\xi_0) \times L} \times \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T_-(J)},$$

such that: (1) given $(p^T, q^T, \alpha^T, m^T)$, for every consumer $h \in H$, $y^{h,T}$ is an optimal solution for $P^{h,T}$, (2) physical and financial markets clear; and (3) there exists perfect foresight for both anonymous rates of payment and total amounts of default.

Equilibria in truncated economies. In order to show existence of equilibria in $\mathcal{E}^T$, we consider generalized games as follows. Given $(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta) \in \mathbb{F}^T := \mathbb{E}^T \times \mathbb{R}_{++}^{D^T(J)} \times \mathbb{R}_{++}^{D^T(\xi_0) \setminus \{\xi_0\}}$, let $\mathcal{K}(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}) = [0, \mathcal{X}] \times [0, \Theta] \times [0, \Psi] \times [0, \tilde{\Psi}]$ be a compact subset of $\mathbb{E}^T$ and let

$$\mathbb{P}_{Q,\beta}^T = \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^L \times [0, Q_\xi] \right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^L \times \prod_{(\mu,j) \in D^T_-(J)} [\underline{\alpha}_{\mu,j}, 1] \times \prod_{\mu \in D^T(\xi_0) \setminus \{\xi_0\}} [0, \beta_\mu].$$

Let $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$ be a generalized game where each consumer is represented by a player $h \in H$ and, at each $\xi \in D^T(\xi_0)$, there are also two players who behave as auctioneers.

More precisely, in $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$ each player $h \in H$ behaves as price-taker and, given $(p, q, \alpha, m) \in \mathbb{P}_{Q,\beta}^T$, she chooses strategies in the truncated budget set $B^{h,T}(p, q, \alpha, m) \cap \mathcal{K}(\mathcal{X}, \Theta, \Psi, \tilde{\Psi})$ in order to maximize the utility function $U^{h,T} := \sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi))$. Also, at each $\xi \in D^{T-1}(\xi_0)$ (resp. $\xi \in D_T(\xi_0)$) one of the corresponding auctioneers chooses commodity and asset prices $(p(\xi), q(\xi)) \in \Delta_+^L \times [0, Q_\xi]$ (resp. just commodity prices $p(\xi) \in \Delta_+^L$) in order to maximize the function $\sum_{h \in H} g_\xi^{h,T}(y^h(\xi), y^h(\xi^-); p, q, \alpha)$, where $y^h = (y^h(\xi))_{\xi \in D^T(\xi_0)}$ are the strategies selected by player $h \in H$ and the payment rates α are chosen by other auctioneers. In fact, there is a player at each node $\xi \in D^T(\xi_0) \setminus \{\xi_0\}$ who determines the rates of payment $\alpha_\xi = (\alpha_{\xi,j}; j \in J^T(\xi^-))$, and the amount of default m_ξ, by minimizing, in the set of variables $(\alpha'_\xi, m'_\xi) \in \prod_{j \in J^T(\xi^-)} [\underline{\alpha}_{\xi,j}, 1] \times [0, \beta_\xi]$, the following function,

$$\left(\alpha'_j(\xi) \sum_{h \in H} \varphi_j^h(\xi^-) - \sum_{h \in H} \tilde{\varphi}_j^h(\xi) \right)^2 + \left(m'_\xi - \sum_{h \in H} \left(T_\xi(\varphi^h(\xi^-)) - t_\xi(\tilde{\varphi}^h(\xi)) \right) \right)^2.$$

DEFINITION A2. A strategy profile $\left[(p^T(\xi), q^T(\xi)); (y^{h,T}(\xi))_{h \in H} \right]_{\xi \in D^T(\xi_0)} \in \mathbb{P}_{Q,\beta}^T \times \left(\mathcal{K}(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}) \right)^H$ is a *Nash equilibrium* for $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$ if each player maximizes her objective function, given the strategies chosen by the other players, i.e., no player has an incentive to deviate.

LEMMA A1. Let $T \in \mathbb{N}$ and $(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta) \in \mathbb{F}^T$. Under Assumptions A-C the set of Nash equilibria for the game $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$ is non-empty.

PROOF. Each player's strategy set is non-empty, convex and compact. Further, it follows from Assumption B that the objective function of each player is continuous and quasi-concave in her own strategy. Assumptions A and C assure that the correspondences of admissible strategies are continuous, with non-empty, convex and compact values. Therefore, we can find an equilibrium of the generalized game by applying Kakutani Fixed Point Theorem to the correspondence defined as the product of the optimal strategy correspondences.

□

LEMMA A2. Let $T \in \mathbb{N}$. Under Assumptions A-C, there exists $(\Theta^T, \Psi^T, \tilde{\Psi}^T, \beta^T)$ such that, if $(\Theta, \Psi, \tilde{\Psi}, \beta) \gg (\Theta^T, \Psi^T, \tilde{\Psi}^T, \beta^T)$, then every Nash equilibrium of the game $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$ is an equilibrium of the economy $\mathcal{E}^T$ whenever $\mathcal{X}$ and Q are large enough.

PROOF. Let $\left[(p^T, q^T, \alpha^T, m^T); (y^{h,T})_{h \in H}\right]$ be a Nash equilibrium for the game $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$, with allocations given by $y^{h,T}(\xi) = (x^{h,T}(\xi), \theta^{h,T}(\xi), \varphi^{h,T}(\xi), \tilde{\varphi}^{h,T}(\xi))$. Note that, for each $h \in H$,

$$(y^{h,T}(\xi))_{\xi \in D^T(\xi_0)} \in \mathrm{argmax}_{B^{h,T}(p^T, q^T, \alpha^T, m^T) \cap \mathcal{K}(\mathcal{X}, \Theta, \Psi, \tilde{\Psi})} \ U^{h,T}(x).$$

Then, it follows that $\sum_{h \in H} g_\xi^{h,T}(y^{h,T}(\xi), y^{h,T}(\xi^-); p^T, q^T, \alpha^T) \leq 0$. As there exists an auctioneer who chooses prices to maximize this previous function, we have that, at each $\xi \in D^T(\xi_0)$,

$$\sum_{h \in H} x^{h,T}(\xi) \leq \Upsilon^T(\Theta, \xi) := \sum_{h \in H} \left(w^h(\xi) + \sum_{j \in J^T(\xi^-)} A(\xi, j)\Theta(\xi^-, j) \right).$$

It follows from Assumptions B that, for each $\xi \in D^T(\xi_0)$, there exists a real number $a_\Theta^T(\xi) > 0$ such that,

$$\min_{h \in H} \ u^h \left(\xi, (a_\Theta^T(\xi), \ldots, a_\Theta^T(\xi)) \right) > \max_{h \in H} \ U^{h,T}(\Upsilon^T(\Theta)),$$

where $\Upsilon^T(\Theta) := (\Upsilon^T(\Theta, \xi); \xi \in D^T(\xi_0))$.

Suppose that $\mathcal{X}(\xi, l) > a_\Theta^T(\xi)$, for every $(\xi, l) \in D^T(\xi_0) \times L$. As $\|p^T(\xi)\|_\Sigma = 1$, it follows from individual optimality that the value of individual financial endowments, at any $\xi \in D^T(\xi_0)$, is necessarily less than $p^T(\xi)(a_\Theta^T(\xi), \ldots, a_\Theta^T(\xi)) = a_\Theta^T(\xi)$. Therefore, for each $j \in J^T(\xi)$,

$$q_j^T(\xi) \leq Q_\Theta^T(\xi, j) := \frac{a_\Theta^T(\xi) \, \#H}{\sum_{h \in H} e_j^h(\xi)}.$$

Let $Q_\Theta^T = (Q_\Theta^T(\xi, j); (\xi, j) \in D^T(J))$. We conclude that if $Q \gg Q_\Theta^T$ then, at any Nash equilibrium of $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$, the upper bounds of asset prices are non-binding. Along the rest of this proof we assume that this property holds.

Step 1. Physical markets clear. We define, node by node, upper bounds for the aggregated resources as follows, $W(\xi_0) = \sum_{h \in H} w^h(\xi_0)$ and $W(\xi) = \sum_{h \in H} \left(w^h(\xi) + A(\xi, j) \sum_{j \in J(\xi^-)} e_j^h(\xi^-) \right)$, for any $\xi > \xi_0$. Now, budget feasibility of individual allocations implies that

$$(1) \qquad \alpha_j^T(\xi) \sum_{h \in H} \varphi_j^{h,T}(\xi^-) = \sum_{h \in H} \tilde{\varphi}_j^{h,T}(\xi), \ \ \forall \xi \in D^T(\xi_0) \setminus \{\xi_0\}, \forall j \in J(\xi^-).$$

Define,

$$\Gamma(\xi_0) \ = \ \sum_{h \in H} \left(x^{h,T}(\xi) - w^h(\xi) \right),$$

$$\Gamma(\xi) \ = \ \sum_{h \in H} \left(x^{h,T}(\xi) - w^h(\xi) - \alpha_j^T(\xi)A(\xi, j) \sum_{j \in J(\xi^-)} e_j^h(\xi^-) \right), \quad \forall \xi \in D^T(\xi_0) \setminus \{\xi_0\};$$

$$\Omega(\xi) \ = \ \sum_{h \in H} \theta^{h,T}(\xi) - \sum_{h \in H} e^h(\xi) - \sum_{h \in H} \varphi^{h,T}(\xi), \quad \forall \xi \in D^T(\xi_0).$$

Summing up the budget constraints at ξ_0 we have $p^T(\xi_0)\Gamma(\xi_0) + q^T(\xi_0)\Omega(\xi_0) \leq 0$. Since one of the auctioneers at ξ_0 maximizes $p(\xi_0)\Gamma(\xi_0)+q(\xi_0)\Omega(\xi_0)$, we obtain that $\Gamma(\xi_0) \leq 0$. Assume now that $\Omega(\xi_0,j) > 0$, for some $j \in J^T(\xi_0)$. By the construction of the plan Q, we know that $q_j^T(\xi_0) < Q_{\xi_0,j}$, which leads us to obtain a contradiction with the optimal behaviour of the auctioneer at ξ_0. Thus $\Omega(\xi_0) \leq 0$. Hence, if $\mathcal{X}(\xi_0,l) > \max\{W(\xi_0,l),a_\Theta^T(\xi_0)\}$ for each $l \in L$, then the upper bound on consumption is not binding at ξ_0, allowing us to conclude, as a consequence of the monotonicity of preferences, that commodity markets clear at the initial node ξ_0, i.e. $\Gamma(\xi_0) = 0$. Moreover, $q^T(\xi_0)\Omega(\xi_0) = 0$.

Consider now a node ξ with $t(\xi) = 1$, and recall that the corresponding auctioneer at ξ chooses prices in $\Delta_+^L \times [0,Q_\xi]$ in order to maximize the function $\sum_{h \in H} g_\xi^{h,T}(y^{h,T}(\xi), y^{h,T}(\xi); p, q, \alpha^T)$. Using the fact that $\Omega(\xi_0) \leq 0$, we can deduce from equation (1), that $p^T(\xi)\Gamma(\xi) + q^T(\xi)\Omega(\xi) \leq 0$, for every ξ with $t(\xi) = 1$. As before, $\Gamma(\xi) \leq 0$ and $\Omega(\xi) \leq 0$. Furthermore, if $\mathcal{X}(\xi) > \max\{W(\xi,l),a_\Theta^T(\xi)\}$ for every $l \in L$, then the upper bound on consumption is not binding at ξ, which implies that $\Gamma(\xi) = 0$.

By applying successively analogous arguments to the nodes with periods $t = 2,\ldots,T$, we conclude that $\Gamma(\xi) = 0$ for every $\xi \in D^T(\xi_0)$, provided that, for each $l \in L$, $\mathcal{X}(\xi,l) > \max\{W(\xi,l), a_\Theta^T(\xi)\}$. That is, physical markets clear in the economy $\mathcal{E}^T$. Furthermore, there is no excess of demand for financial markets, i.e. $\Omega(\xi) \leq 0$, for every $\xi \in D^{T-1}(\xi_0)$.

Step 2. Lower bounds of asset prices. Given $(\xi,j) \in D^T(J)$, fix a node $\mu(\xi,j) \in \xi^+$ such that $A(\mu(\xi,j),j) \neq 0$. It follows from Assumptions A and B that there exists $b(\xi,j) \in (0,1)$, which is independent of T, such that, for every $h \in H$, the following inequality holds,

$$(2) \qquad u^h\left(\mu(\xi,j), w^h(\mu(\xi,j)) + \frac{\underline{\alpha}_{\mu(\xi,j),j}A(\mu(\xi,j),j)\min_{l \in L} w_l^h(\xi)}{b(\xi,j)}\right) > U^h(W).$$

Suppose that,

$$\Theta(\xi,j) > \widehat{\Theta}(\xi,j) := \max_{h \in H} \frac{\min_{l \in L} w_l^h(\xi)}{b(\xi,j)},$$

and for every $\mu \in D^{T-t(\xi)}(\xi)$ with $j \in J^T(\mu)$,

$$\min_{l \in L} \mathcal{X}(\mu,l) > \mathcal{X}_{\Theta,\xi}^T(\mu,j) := \max_{(l,h) \in H \times L}\left\{W(\mu,l), a_\Theta^T(\mu), w_l^h(\mu) + \frac{\underline{\alpha}_{\mu(\xi,j),j}A_l(\mu,j)\min_{l' \in L} w_{l'}^h(\xi)}{b(\xi,j)}\right\}.$$

Then, it is not hard to show that $q_j^T(\xi) > b(\xi,j)$.[4] Therefore, if for each $\eta \in D^T(\xi_0)$,

$$\Theta(\eta,j) > \widehat{\Theta}(\eta,j), \ \forall j \in J^T(\eta),$$
$$\mathcal{X}(\eta,l) > \mathcal{X}_\Theta^T(\eta) := \max_{\substack{(\xi,j) \in D^T(J): \\ \eta > \xi, \ j \in J^T(\eta)}} \mathcal{X}_{\Theta,\xi}^T(\eta,j), \ \forall l \in L,$$

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[4]If $q_j^T(\xi) \leq b(\xi,j)$ then, as by Step 1 $x^{h,T}(\mu) \leq W(\mu)$ for every $\mu \in D^T(\xi_0)$, it follows from Assumption B and equation (2) that any $h \in H$ has an incentive to deviate by choosing any budget feasible strategy $(x^h, \theta^h, \varphi^h, \tilde{\varphi}^h)$ that satisfies,

$$\theta_j^h(\xi) = \frac{\min_{l \in L} w_l^h(\xi)}{b(\xi,j)} \quad \text{and}$$
$$x^h(\mu) = w^h(\mu) + \underline{\alpha}_{\mu(\xi,j),j}A(\mu,j)\theta_j^h(\xi), \quad \text{if } \mu = \mu(\xi,j).$$

then equilibrium asset prices have a positive lower bound away from zero. In fact, for each $(\eta, j) \in D^T(J)$, we have that $q_j^T(\eta) > b(\eta, j)$.

Step 3. Non-binding short-sales constraints. Define $\widehat{\Theta}^T = (\widehat{\Theta}(\eta, j); (\eta, j) \in D^T(J))$ and $\mathcal{X}_\Theta^T = (\mathcal{X}_\Theta^T(\eta); \eta \in D^T(\xi_0))$. If $\Theta \gg \widehat{\Theta}^T$ and $\mathcal{X} \gg \mathcal{X}_\Theta^T$, then asset prices are bounded away from zero. Thus, using the borrowing constraints, we conclude that, for every player $h \in H$,

$$\varphi_j^{h,T}(\xi) < \widehat{\Psi}_j(\xi) := \overline{C} \frac{\max_{l \in L} M_l}{b(\xi, j)}, \qquad \forall (\xi, j) \in D^T(J),$$

where $\overline{C}$ is an uniform upper bound for the credit constraint functions $\mathbb{C}_\mu^h$. Let $\Psi^T = (\widehat{\Psi}_j(\xi); (\xi, j) \in D^T(J))$. Given $(\xi, j) \in D^T(J)$, define $\tilde{\Psi}_j^T(\mu) = \Psi_j^T(\xi)$, for each node $\mu \in \xi^+$. If $(\Psi, \tilde{\Psi}) \gg (\Psi^T, \tilde{\Psi}^T)$, then the restrictions in short-sales induced by $\mathcal{K}(\mathcal{X}, \Theta, \Psi, \tilde{\Psi})$ are not binding.

Step 4. Financial markets clear and upper bounds for long-positions are non-binding. Suppose that $(\Theta, \Psi) \gg (\widehat{\Theta}^T, \Psi^T)$ and $\mathcal{X} \gg \mathcal{X}_\Theta^T$. Now, by Step 1 we have that $q^T(\xi)\Omega(\xi) = 0$ and $\Omega(\xi) \leq 0$, for each $\xi \in D^{T-1}(\xi_0)$. Thus, if for some $(\xi, j) \in D^T(J)$, $\Omega_j(\xi) < 0$, then $q_j^T(\xi) = 0$, which is in contradiction with the lower bound on asset prices find in Step 2. Therefore, financial markets feasibility holds and, by equation (1), the equilibrium condition regarding perfect foresight on anonymous rates of payments is also true.

On the other hand, for each $\xi \in D^{T-1}(\xi_0)$, $(\varphi^{h,T}(\xi))_{h \in H}$ is bounded. Thus, as $\Omega(\xi) \leq 0$, $\sum_{h \in H} \theta^{h,T}(\xi)$ is also bounded. We conclude that there exists $\Theta^T \geq \widehat{\Theta}^T$ such that, if $\Theta \gg \Theta^T$ then upper bounds on long positions are non-binding.

Step 5. Individual perfect foresight the total amount of default. Note that short sales are bounded by the plan Ψ^T. Then, for every node ξ, there exists $\overline{\beta}(\xi)$ such that the auctioneer who selects the amount of default at ξ can get zero payoff whenever $\beta \gg \beta^T := (\overline{\beta}(\mu); \mu \in D^T(\xi_0) \setminus \xi_0)$. Thus, the condition on default perfect foresight, which is required at equilibrium, holds.

Step 6. Individual optimality. As a consequence of all previous steps, if $(\Theta, \Psi, \tilde{\Psi}) \gg (\Theta^T, \Psi^T, \tilde{\Psi}^T)$ and $(\mathcal{X}, Q) \gg (\mathcal{X}_\Theta^T, Q_\Theta^T)$ then, for each $h \in H$, the optimal allocation $y^{h,T}$ belongs to the interior of $\mathcal{K}(\mathcal{X}, \Theta, \Psi, \tilde{\Psi})$ (relative to $\mathbb{E}^T$). As budget correspondences has finite-dimensional convex values, we conclude that $(y^{h,T}(\xi))_{\xi \in D^T(\xi_0)}$ maximize $\sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi))$ on $B^{h,T}(p^T, q^T, \alpha^T, m^T)$.

Therefore, any Nash equilibrium of $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, \tilde{\Psi}, Q, \beta)$ is an equilibrium of $\mathcal{E}^T$, provided that $(\Theta, \Psi, \tilde{\Psi}, \beta) \gg (\Theta^T, \Psi^T, \tilde{\Psi}^T, \beta^T)$ and $(\mathcal{X}, Q) \gg (\mathcal{X}_\Theta^T, Q_\Theta^T)$. $\qquad \square$

By construction, the upper bounds $(\Theta^T(\xi), \Psi^T(\xi), \beta^T(\xi))$ are independent of $T > t(\xi)$, when T is large enough. Therefore, node by node, independently of the truncated horizon T, individual equilibrium allocations are uniformly bounded and commodity prices belong to the simplex.

Moreover, under Assumptions B and C, asset prices are uniformly bounded by above, node by node. In fact, as consumption allocations are bounded by the aggregated resources (which are bounded by $W = (W(\xi); \xi \in D)$) by analogous arguments to those made in the proof of Lemma A2, we can conclude that, $q_j^T(\xi) \leq \frac{a(\xi) \#H}{\sum_{h \in H} e_j^h(\xi)}, \forall j \in J^T(\xi)$, where $a(\xi) > 0$ is independent of $T > t(\xi)$ and is defined

implicitly by $\min_{h \in H} \; u^h \left(\xi, (a(\xi), \ldots, a(\xi))\right) > \max_{h \in H} U^h(W)$.

ASYMPTOTIC EQUILIBRIA. We look for an uniform bound (node by node) for the Kuhn-Tucker multipliers associated to the truncated individual problems. Thus, for each $T \in \mathbb{N}$, consider an equilibrium $\left[(p^T, q^T, \alpha^T, m^T); (y^{h,T}(\xi))_{h \in H}\right]_{\xi \in D^T(\xi_0)}$ for $\mathcal{E}^T$. Then, there exist *non-negative* multipliers $\left((\gamma_\xi^{h,T})_{\xi \in D^T(\xi_0)}; (\rho_\xi^{h,T})_{\xi \in D^{T-1}(\xi_0)}\right)$ such that,

$$(3) \qquad\qquad \gamma_\xi^{h,T} g_\xi^{h,T}(y^{h,T}(\xi), y^{h,T}(\xi^-); p^T, q^T, \alpha^T) \;=\; 0, \quad \forall \xi \in D^T(\xi_0);$$

$$(4) \qquad \rho_\xi^{h,T}\left(\mathbb{C}_\xi^h\left[T_\xi(\varphi^h(\xi^-)),\; t_\xi(\tilde\varphi^h(\xi)),\; m_\xi^T\right] p^T(\xi)M - q^T(\xi)\varphi^{h,T}(\xi)\right) \;=\; 0, \quad \forall \xi \in D^{T-1}(\xi_0).$$

Moreover, for each plan $(x(\xi), \theta(\xi), \varphi(\xi), \tilde\varphi(\xi))_{\xi \in D^T(\xi_0)} \geq 0$, with $(\theta(\eta), \varphi(\eta), \tilde\varphi(\eta))_{\eta \in D_T(\xi_0)} = 0$, the following saddle point property is satisfied (see Rockafellar (1997), Section 28, Theorem 28.3),

$$(5) \qquad U^{h,T}(x) - \sum_{\xi \in D^T(\xi_0)} \gamma_\xi^{h,T} g_\xi^{h,T}(y(\xi), y(\xi^-); p^T, q^T, \alpha^T)$$

$$+ \sum_{\xi \in D^{T-1}(\xi_0)} \rho_\xi^{h,T}\left(\mathbb{C}_\xi^h\left[T_\xi(\varphi^h(\xi^-)),\; t_\xi(\tilde\varphi^h(\xi)),\; m_\xi^T\right] p^T(\xi)M - q^T(\xi)\varphi^{h,T}(\xi)\right) \leq U^{h,T}(x^{h,T})$$

Let us take $(x(\xi), \theta(\xi), \varphi(\xi), \tilde\varphi(\xi))_{\xi \in D^T(\xi_0)} = (0,0,0,0)$ to obtain,

$$(6) \qquad\qquad \sum_{\xi \in D^{T-1}(\xi_0)} \left[\gamma_\xi^{h,T} p^T(\xi) w^h(\xi) + \rho_\xi^{h,T} \tau(\xi) p^T(\xi)M\right] \leq U^h(W) < +\infty.$$

As commodity prices are in the simplex, node by node, and $M \in \mathbb{R}_{++}^L$, using Assumptions A and C we conclude that, at each node $\xi \in D$, the sequence formed by equilibrium prices, equilibrium allocations and Kuhn-Tucker multipliers, $((p^T(\xi), q^T(\xi), \alpha^T(\xi), m^T(\xi)); (y^{h,T}(\xi), \gamma_\xi^{h,T}, \rho_\xi^{h,T})_{h \in H})_{T > t(\xi)}$, is bounded. Therefore, applying Tychonoff Theorem we can find a common subsequence converging to an allocation $\left((\overline{p}(\xi), \overline{q}(\xi), \overline{\alpha}(\xi), \overline{m}(\xi)); (\overline{y}^h(\xi), \overline{\gamma}_\xi^h, \overline{\rho}_\xi^h)_{h \in H}\right)$.

Therefore, since market clearing follows, it remains to show that, for each $h \in H$, $(\overline{y}^h(\xi))_{\xi \in D}$ is an optimal in $B^h(\overline{p}, \overline{q}, \overline{\alpha}, \overline{m})$.

LEMMA A3. *Under Assumptions A-C, $U^h(x) \leq U^h(\overline{x})$, for every allocation $(x, \theta, \varphi, \tilde\varphi) \in B^h(\overline{p}, \overline{q}, \overline{\alpha}, \overline{m})$.*

PROOF. Given $(x, \theta, \varphi, \tilde\varphi) \in B^h(\overline{p}, \overline{q}, \overline{\alpha}, \overline{m})$ and following the same arguments as those used in Lemma A3 in Moreno-García and Torres-Martínez (2006), it follows that, for each $N \in \mathbb{N}$,

$$U^{h,N}(x) - U^{h,N}(\overline{x}) \leq \sum_{\mu \in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h g_\mu^h(\overline{y}^h(\mu), \tilde{y}(\mu^-); \overline{p}, \overline{q}, \overline{\alpha}).$$

Thus, borrowing constraints imply that,

$$(7) \qquad U^{h,N}(x) - U^{h,N}(\overline{x}) \leq \sum_{\mu \in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h \left(\overline{p}(\mu)\overline{x}^h(\mu) + \overline{q}(\mu)(\overline{\theta}^h(\mu) - \overline{\varphi}^h(\xi)) + \overline{C}\,\overline{p}(\mu)M\right).$$

Also, as in Moreno-García and Torres-Martínez (2006), this implies that,

$$U^{h,N}(x) - U^{h,N}(\overline{x}) \leq \sum_{\mu \in D \setminus D^N(\xi_0)} u^h(\mu, W(\mu)) + \overline{C} \sum_{\mu \in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h \overline{p}(\mu)M.$$

Now, Assumption A and inequality (6) guarantee that,

$$(8) \qquad \sum_{\xi \in D} \overline{\gamma}_{\xi}^{h} \|\overline{p}(\xi)\|_{\Sigma} < +\infty.$$

Therefore, it follows from Assumption B that: For each $\varepsilon > 0$ there exists $\overline{N}_{\varepsilon} > 0$ such that,

$$\sum_{\xi \in D^{N}(\xi_0)} u^{h}(\xi, x(\xi)) < \varepsilon + U^{h}(\overline{x}), \quad \forall N > \overline{N}_{\varepsilon}$$

Finally, we conclude that, for each $\varepsilon > 0$, $U^{h}(x) \leq \varepsilon + U^{h}(\overline{x})$, which ends the proof. □

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