TEXTO PARA DISCUSSÃO

No. 541

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WELFARE-IMPROVING DEBT CONSTRAINTS

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Abstract. Under uniform impatience of preferences, assets in positive net supply are free of price bubbles for deflators that yield finite present values of wealth. However, this does not imply that equilibrium prices must coincide with present values of dividends. Indeed, if borrowing constraints become binding, asset prices must take into account the respective shadow prices.

In this context, we analyze the widely studied case of an asset paying no dividends where loans are bounded by an explicit debt constraint. We prove that a positive asset price occurs at some node if and only debt constraints are binding at this node or at some future state of nature. Thus, binding debt constraints always induce frictions which create room for improving welfare by allowing money to have a role in transferring wealth across the event tree.

Keywords: Binding debt constraints, Fundamental value, Rational asset pricing bubbles.

JEL classification: D50, D52.

1. Introduction

Long-lived assets in positive net supply, such as equity and fiat money, have been extensively studied in two sorts of general equilibrium models, resulting in two quite different conclusions with regard to how equilibrium prices are related to the series of discounted dividends. In overlapping generations models, prices may exceed fundamental values, but models with infinite-lived agents have been hostile to bubbles (see Magill-Quinzii (1996) and Santos-Woodford (1997)). These different results have to do with the fact that in the former the present value of wealth may be infinite, whereas in the latter it must be finite, at least for deflator processes generated by the Kuhn-Tucker multipliers. It is precisely for these deflators that, under uniform impatience, assets in positive net supply are free of price bubbles.

The uniform impatience assumption (see Hernandez and Santos (1996) or Magill and Quinzii (1996)) is a usual requirement for existence of equilibrium in economies with infinite lived debt-constrained agents. It holds for time and state separable preferences, provided that inter-temporal discounted factor is constant, individual endowments are uniformly bounded away from zero and aggregate endowments are uniformly bounded from above. In this paper, we show that uniform impatience is still compatible with assets in positive net supply being priced in excess of the discounted stream of dividends. The necessary and sufficient condition for this pricing deviation to occur is that each agent has binding debt constraints at some future node.

Revised version: December, 2007.

A previous version of this discussion paper appeared under the title "On the role of debt constraints in monetary equilibrium". J.P.Torres-Martínez acknowledges support from CNPq through project 307554/2004-0.

* M. Petrassi wants to disclaim that the views expressed herein are not necessarily those of the Central Bank of Brazil. *As visões expressas no trabalho são de exclusiva responsabilidade dos autores.*

We focus on the widely addressed case of an asset paying no dividends and with positive endowments at the initial node. This case was examined by Samuelson (1958) in the overlapping generations context and, then, by Bewley (1980) and many other authors in the context of infinite-lived households. This benchmark case is both simpler and more intriguing. As usual, we call this asset fiat money, although we are quite aware that we are just looking at its role as a store of value, i.e. as an instrument to transfer wealth across time and states of nature. We are abstracting from the two other roles of money, that is, medium of exchange and unit of account.[1] In our model, the frictions that will be responsible for a positive price of money are credit frictions, that is, binding credit ceilings that have to be imposed in order to avoid Ponzi schemes. In models addressing the role of money as a medium of exchange, starting with Clower (1967), it is instead liquidity frictions that become crucial. In a recent work along those lines, Santos (2006) showed that monetary equilibrium only arises when cash-in-advance constraints are binding infinitely often for all agents. Here, we contemplate a pure credit economy where money can still be positively valued as a result of agents' desire to take monetary loans when they cannot (either because monetary loans are not allowed or because a debt ceiling has been hit).[2]

Since shadow prices of debt constraints will play a crucial role, we develop a duality theory for the households' dynamic programming problem. We identify the Euler and transversality conditions that characterize individual optimality and show that, under the Kuhn-Tucker multipliers, the present value of endowments must be always finite. Once we combine this property with the uniform impatience property we rule out bubbles in the price of money and, therefore, a positive price of money can only occur under binding debt constraints. Hence, we obtain a result that may seem surprising: credit frictions create room for welfare improvements through inter-temporal and inter-states transfers of wealth that become possible when money has a positive price.

It is important to notice that the above monetary equilibrium is always Pareto inefficient. Otherwise, the agents' rates of inter-temporal substitution would coincide but, as money is in positive net supply, at least one agent must go long, having a zero shadow price. Thus, the shadow prices of all agents should be zero and, therefore, the price of money could not be positive.

When money has a positive value, there exists a deflator, but not one of the Kuhn-Tucker deflators, under which the discounted value of aggregated wealth is infinite and a pure bubble appears. Also, independently of the non-arbitrage deflator, when aggregated endowments can be replicated by a portfolio trading plan, the discounted value of future wealth must be finite (see Santos and Woodford (1997)). Therefore, if we allow for an increasing number of non-redundant securities in order to assure that aggregated wealth can be replicated by the deliveries of a portfolio trading plan, money will have zero price. However, the issue of new assets, in order to achieve that efficacy of the financial markets, can be too costly.

Finally, we show that uniform impatience of preferences is fundamental to our results. In fact, we provide an example in which utility functions do not satisfy uniform impatience and allow for

[1]Perhaps some readers may argue that these others roles are more important. However, we do have the objective to address the delicate issue of the essentiality of money.

[2]In a similar context, Gimenez (2005) provided examples of monetary bubbles that can be reinterpreted as positive fundamental values in cashless economies with *no short-sales* restrictions.

speculation in an asset in positive net supply, even for deflators that yield finite present values of wealth.

The rest of the paper is organized as follows. Section 2 presents the basic model. In Section 3, our results are proved. In the Appendix A we develop the necessary mathematical tools: a duality theory of individual optimization. Other important results are proved in Appendices B and C.

2. Model

We consider an infinite horizon discrete time economy. The set of dates is $\{0,1,\dots\}$ and there is no uncertainty at $t=0$. However, given a history of realizations of the states of nature for the first $t-1$ dates, with $t \geq 1$, $\overline{s}_t=(s_0,\dots,s_{t-1})$, there is a finite set $S(\overline{s}_t)$ of states of nature that may occur at date t. A vector $\xi=(t,\overline{s}_t,s)$, where $t\geq 1$ and $s\in S(\overline{s}_t)$, is called a *node* of the economy. The only node at $t=0$ is denoted by ξ_0. Let D be the *event-tree*, i.e., the set of all nodes.

Given $\xi=(t,\overline{s}_t,s)$ and $\mu=(t',\overline{s}_{t'},s')$, we say that μ is a *successor* of ξ, and we write $\mu\geq\xi$, if $t'\geq t$ and $\overline{s}_{t'}=(\overline{s}_t,s,\dots)$. We write $\mu>\xi$ to say that $\mu\geq\xi$ but $\mu\neq\xi$ and we denote by $t(\xi)$ the date associated with a node ξ. Let $\xi^+=\{\mu\in D:(\mu\geq\xi)\wedge(t(\mu)=t(\xi)+1)\}$ be the set of immediate successors of ξ. The (unique) predecessor of ξ is denoted by ξ^- and $D(\xi):=\{\mu\in D:\mu\geq\xi\}$ is the sub-tree with root ξ.

At each node, a finite set of perishable commodities is available for trade, L. Let $p=(p(\xi);\xi\in D)$, where $p(\xi):=(p(\xi,l);l\in L)$ denotes the commodity price at $\xi\in D$. We assume that there is only one asset, *money*, that can be traded at any node along the event-tree. Although this security does not deliver any payment, it can be used to make inter-temporal transfers. Let $q=(q(\xi);\xi\in D)$ be the plan of state-dependent monetary prices. We assume that money has positive net supply that does not disappear from the economy neither depreciate.

A finite number of agents, $h\in H$, can trade money and buy commodities along the event-tree. Agent h is characterized by his physical and financial endowments, $(w^h(\xi),e^h(\xi))\in\mathbb{R}^L_{++}\times\mathbb{R}_+$, at each $\xi\in D$, and by his preferences over consumption, which are represented by an utility function $U^h:\mathbb{R}^{D\times L}_+\to\mathbb{R}_+\cup\{+\infty\}$. For any $\xi\in D$, let $W_\xi=\sum_{h\in H}w^h(\xi)$ be the aggregated physical endowment at node ξ.

The consumption allocation of agent h at $\xi\in D$ is denoted by $x^h(\xi):=(x^h(\xi,l);l\in L)$. Analogously, the number $z^h(\xi)$ denotes the quantity of money that h negotiates at ξ. Thus, if $z^h(\xi)>0$, he buys the asset, otherwise, he short sales money making future promises.

Given prices (p,q), let $B^h(p,q)$ be the choice set of agent $h\in H$, that is, the set of plans $(x,z):=((x(\xi),z(\xi));\xi\in D)\in\mathbb{R}^{D\times L}_+\times\mathbb{R}^D$, such that, at each $\xi\in D$, the following budget and debt constraint hold,

$$\begin{aligned} g^h_\xi(y^h(\xi),y^h(\xi^-);p,q):=p(\xi)\left(x^h(\xi)-w^h(\xi)\right)+q(\xi)\left(z^h(\xi)-e^h(\xi)-z^h(\xi^-)\right) &\leq 0,\\ q(\xi)z^h(\xi)+p(\xi)M &\geq 0,\end{aligned}$$

where $y^h(\xi)=(x^h(\xi),z^h(\xi))$, $z^h(\xi_0^-)=0$ and $M\in\mathbb{R}^L_+$. Note that short sales of money are bounded by the exogenous *debt constraints* above in order to avoid Ponzi schemes. Agent's h individual problem is to choose a plan $y^h = (x^h, z^h)$ in $B^h(p,q)$ in order to maximize his utility functions U^h.

DEFINITION 1. *An equilibrium for our economy is given by a vector of prices (p,q) jointly with individual allocations $\big((x^h, z^h); h \in H\big)$, such that,*

(a) *For each $h \in H$, the plan $(x^h, z^h) \in B^h(p,q)$ is optimal, at prices (p,q),*
(b) *Physical and asset markets clear,*

$$\sum_{h\in H} \left(x^h(\xi); z^h(\xi)\right) = \left(W_\xi, \sum_{h\in H} \left(e^h(\xi) + z^h(\xi^-)\right)\right).$$

3. Characterizing monetary equilibria

In our economy, a *pure spot market equilibrium*, i.e. an equilibrium with zero monetary price, always exists provided that preferences satisfy he following hypothesis.

ASSUMPTION A. *For any agent $h \in H$, $U^h(x) := \sum_{\xi\in D} \; u^h(\xi, x(\xi))$, where for any ξ, the function $u^h(\xi, \cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ is concave, continuous and strictly increasing. Also, $\sum_{\xi\in D} u^h(\xi, W_\xi)$ is finite.*

However, our objective is to determine conditions that characterize the existence of equilibria with positive price of money, also called *monetary equilibria*. For this reason, we assume that agents are *uniformly impatience*.

ASSUMPTION B. *There are $\pi \in [0,1)$ and $(v(\mu); \mu \in D) \in \mathbb{R}^{D\times L}_+$ such that, for each $h \in H$, given a consumption plan $(x(\mu); \mu \in D)$, with $0 \le x(\mu) \le W_\mu$, we have that,*

$$u^h\left(\xi, x(\xi) + v(\xi)\right) + \sum_{\mu>\xi} u^h(\mu,\, \pi' \, x(\mu)) > \sum_{\mu\ge\xi} u^h(\mu, x(\mu)), \qquad \forall \xi \in D,\;\; \forall \pi' \ge \pi.$$

Moreover, there is $\delta > 0$ such that, $w^h(\xi) \ge \delta v(\xi)$, $\forall \xi \in D$.

The requirements of impatience above depend on both preferences and physical endowments. As particular cases we obtain the assumptions imposed by Hernandez and Santos (1996) and Magill and Quinzii (1996). Indeed, in Hernandez and Santos (1996), for any $\mu \in D$, $v(\mu) = W_\mu$. Also, since in Magill and Quinzii (1996) initial endowments are uniformly bounded away from zero by an interior bundle $\underline{w} \in \mathbb{R}^L_+$, they suppose that $v(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Under Assumption A, Propositions A1 and B1 in the Appendices assure that, given an equilibrium $\big[(p,q); ((x^h, z^h); h \in H)\big]$, there are, for each $h \in H$, Kuhn-Tucker multipliers $(\gamma^h(\xi); \xi \in D)$, such that,

$$q(\xi) = F(\xi, q, \gamma^h(\xi)) + \lim_{T\to+\infty} \sum_{\{\mu\ge\xi : t(\mu)=T\}} \frac{\gamma^h(\mu)}{\gamma^h(\xi)} q(\mu),$$

where $F(\xi, q, \gamma^h(\xi))$ is the *fundamental value* of money, and the second term in the right hand side is the monetary speculative component, also called *bubble*. We say that *debt constraints induce* *frictions over agent h in $\tilde{D} \subset D$* if the plan of shadow prices $(\eta^h(\mu);\, \mu \in \tilde{D})$ that is defined implicitly, at each $\mu \in \tilde{D}$, by the conditions:

$$\begin{aligned} 0 &= \eta^h(\mu)\left(q(\mu)z^h(\mu) + p(\mu)M\right), \\ \gamma^h(\mu)q(\mu) &= \sum_{\nu \in \mu^+} \gamma^h(\nu)q(\nu) + \eta^h(\mu)q(\mu), \end{aligned}$$

is different from zero.

Theorem. *Under Assumptions A-B, for any equilibrium $\left[(p,q);((x^h,z^h);h \in H)\right]$ we have that,*

(1) *If $q(\xi) > 0$ then debt constraints induce frictions over each agent in $D(\xi)$.*
(2) *If $M \neq 0$ and some $h \in H$ has a binding debt constraint at a node $\mu \in D(\xi)$, then $q(\xi) > 0$.*
(3) *If for each $\xi \in D$, $u^h(\xi,\cdot)$ is differentiable in $\mathbb{R}^L_{++}$ and $\lim_{\|x\|_{min} \to 0^+} \nabla u^h(\xi,x) = +\infty$, then any monetary equilibrium is Pareto inefficient.*

Observation. Item (1) is related to the result in Santos and Woodford (1997), Theorem 3.3, that asserted that, under uniform impatience, assets in positive net supply are free of price bubbles for deflators that yield finite present values of wealth. However, in the frictionless framework used by these authors, absence of bubbles led necessarily to a zero price of money. The converse, item (2), and item (3) were not addressed before.

Moreover, it follows from items (1) and (2) that binding debt constraints always induce frictions, i.e. positive shadow prices. Also, if an agent becomes borrower at some node in $D(\xi)$, then *all individuals* are borrowers at some node of $D(\xi)$. In other words, in a monetary equilibrium, all agents take a monetary loan (at some node).

Proof of the Theorem. (1) By definition, if for some $h \in H$, $(\eta^h(\mu);\mu \geq \xi) = 0$ then $F(\xi,q,\gamma^h(\xi)) = 0$. Therefore, as in Santos and Woodford (1997), a monetary equilibrium is a *pure bubble*. However, Assumption B implies that bubbles are ruled out in equilibrium.

Indeed, at each $\xi \in D$ there exists an agent $h = h(\xi)$ with $q(\xi)z^h(\xi) \geq 0$. Thus, by the impatience property, $0 \leq (1-\pi)q(\xi)z^h(\xi) \leq p(\xi)v(\xi)$. Moreover, financial market feasibility allows us to find a lower bound for individual debt. Therefore, for each $h \in H$, the plan $\left(\frac{q(\xi)z^h(\xi)}{p(\xi)v(\xi)}\right)_{\xi \in D}$ is uniformly bounded. Furthermore, as money has positive net supply, it follows that $\left(\frac{q(\xi)}{p(\xi)v(\xi)}\right)_{\xi \in D}$ is uniformly bounded too. As by Lemma A1 we know that, for any $h \in H$, $\sum_{\xi \in D} \gamma^h(\xi)p(\xi)w^h(\xi) < +\infty$, it follows from Assumption B that bubbles do not arise in equilibrium.

Therefore, we conclude that, if $q(\xi) > 0$ then $(\eta^h(\mu);\mu \geq \xi) \neq 0$, for all $h \in H$.

(2) Suppose that, for some $h \in H$, there exists $\mu \geq \xi$ such that that $q(\mu)z^h(\mu) = -p(\mu)M$. Since monotonicity of preferences implies that $p(\xi) \gg 0$, if $M \neq 0$ then $q(\mu) > 0$. Also, Assumption A assures that Kuhn-Tucker multipliers, $(\gamma^h(\eta);\eta \in D)$, are strictly positive. Therefore, the equations that define shadow prices implies that $q(\xi) > 0$.

(3) Suppose that there exists an efficient monetary equilibrium, in the sense that individuals' marginal rates of substitution coincide. As $\lim_{\|x\|_{min} \to 0^+} \nabla u^h(\xi,x) = +\infty$, $\forall (h,\xi) \in H \times D$, all

agents have interior consumption along the event-tree. Positive net supply of money implies that there exists, at each $\xi \in D$, at least one lender. Therefore, by the efficiency property, it follows that *all* individuals have zero shadow prices. A contradiction with item (1) above. □

Some remarks,

(1) It follows from the proof of the Theorem that under Assumption B the monetary debt is uniformly bounded—in real terms—along the even-tree. Thus, it is easy to found a vector $M^* \in \mathbb{R}^L_+$ such that, in any equilibrium, and for each node ξ, the debt constraint $q(\xi)z^h(\xi) \geq -p(\xi)M^*$ is non-binding. Therefore, when $M > M^*$ monetary equilibria disappear. That is, contrary to what may be expected, frictions induced by debt constraints improve welfare.

(2) Given a monetary equilibrium, there always exists a non-arbitrage deflator, *incompatible with physical Euler conditions* (see Definition A1), for which the price of money is a pure bubble. Indeed, define $\nu := (\nu(\xi) : \xi \in D)$ by $\nu(\xi_0) = 1$, and

$$\begin{aligned} \nu(\xi) &= 1, & \forall \xi > \xi_0 : q(\xi) = 0, \\ \frac{\nu(\xi)}{\nu(\xi^-)} &= \frac{\gamma^h(\xi)}{\gamma^h(\xi^-) - \eta^h(\xi^-)}, & \forall \xi > \xi_0 : q(\xi) > 0. \end{aligned}$$

Euler conditions on $(\gamma^h(\xi); \xi \in D)$ imply that, for each $\xi \in D$, $\nu(\xi)q(\xi) = \sum_{\mu\in\xi^+} \nu(\mu)q(\mu)$. Therefore, using the plan of deflators ν, *financial Euler conditions* hold and the positive price of money is a bubble. Since under Assumption B the monetary debt is uniformly bounded along the event-tree, under these deflators the discounted value of future individual endowments has to be infinite.

We remark that the plan of state prices ν is compatible with the frictionless theory of bubbles developed by Santos and Woodford (1997) and, in that frictionless context, we recover a property that was previously found by them: a monetary bubble is possible only for deflators under which we have an infinite discounted value of future wealth.

4. About uniform impatience

To highlight the role that uniform impatience has in our Theorem, we adapt Example 1 in Araujo, Páscoa and Torres-Martínez (2007) in order to prove that without Assumption B money may have a pure bubble for Kuhn-Tucker multipliers. Moreover, bubbles on the price of money will be compatible with a finite discounted value of future wealth. Essentially because individuals will believe that, as time goes on, the probability that the economy may fall in a path in which endowments increase without an upper bound converges to zero fast enough.

Example. Assume that each $\xi \in D$ has two successors: $\xi^+ = \{\xi^u, \xi^d\}$. There are only one commodity and two agents $H = \{1, 2\}$. Each $h \in H$ has physical endowments $(w^h_\xi)_{\xi\in D}$, receives financial endowments $e^h \geq 0$ only at the first node, and has preferences represented by the utility function $U^h(x) = \sum_{\xi\in D} \beta^{t(\xi)}\rho^h(\xi)\, x_\xi$, where $\beta \in (0, 1)$ and the plan $(\rho^h(\xi))_{\xi\in D} \in (0, 1)^D$ satisfies

$\rho(\xi_0)=1$, $\rho^h(\xi)=\rho^h(\xi^d)+\rho^h(\xi^u)$ and

$$\rho^1(\xi^u)=\frac{1}{2^{t(\xi)+1}}\rho^1(\xi), \qquad \rho^2(\xi^u)=\left(1-\tfrac{1}{2^{t(\xi)+1}}\right)\rho^2(\xi).$$

Suppose that agent $h=1$ is the only one endowed with the asset, i.e. $(e^1,e^2)=(1,0)$ and that, for each $\xi\in D$,

$$w^1_\xi=\begin{cases}1+\beta^{-t(\xi)} & \text{if } \xi\in D^{du},\\ 1 & \text{otherwise};\end{cases} \qquad w^2_\xi=\begin{cases}1+\beta^{-t(\xi)} & \text{if } \xi\in\{\xi_0^d\}\cup D^{ud},\\ 1 & \text{otherwise};\end{cases}$$

where D^{du} is the set of nodes attained after going down followed by up, that is, $D^{du}=\{\eta\in D:\ \exists\xi,\ \eta=(\xi^d)^u\ \}$ and D^{ud} denotes the set of nodes reached by going up and then down, that is, $D^{ud}=\{\eta\in D:\ \exists\xi,\ \eta=(\xi^u)^d\ \}$.

Agents will use positive endowment shocks in low probability states to buy money and sell it later in states with higher probabilities. Let prices be $(p_\xi,q_\xi)_{\xi\in D}=(\ \beta^{t(\xi)},\,1)_{\xi\in D}$ and suppose that consumption of agent h is given by $x^h_\xi=w^{h'}_\xi$, where $h\neq h'$. It follows from budget constraints that, at each ξ, the portfolio of agent h must satisfy $z^h_\xi=\beta^{t(\xi)}(w^h_\xi-w^{h'}_\xi)+z^h_{\xi^-}$, where $z^h_{\xi_0^-}:=e^h$ and $h\neq h'$.

Thus, the consumption allocations above jointly with the portfolios $(z^1_{\xi_0},\,z^1_{\xi^u},\,z^1_{\xi^d})=(1,\,1,\,0)$ and $(z^2_\xi)_{\xi\in D}=(1-z^1_\xi)_{\xi\in D}$ are budget and market feasible. Finally, given $(h,\xi)\in H\times D$, let $\gamma^h_\xi=\rho^h(\xi)$ be the candidate for Kuhn-Tucker multiplier of agent h at node ξ. It follows that conditions below hold and they assure individual optimality (see Proposition A2 in the Appendix A),

$$\begin{aligned}
(\gamma^h_\xi p_\xi,\gamma^h_\xi q_\xi) &= (\beta^{t(\xi)}\rho^h(\xi),\gamma^h_{\xi^u}q_{\xi^u}+\gamma^h_{\xi^d}q_{\xi^d}),\\
\sum_{\{\eta\in D:\,t(\eta)=T\}}\gamma^h_\eta p_\eta M &\longrightarrow 0, \qquad \text{as } T\to+\infty,\\
\sum_{\{\eta\in D:\,t(\eta)=T\}}\gamma^h_\eta q_\eta z^h_\eta &\longrightarrow 0, \qquad \text{as } T\to+\infty.
\end{aligned}$$

Note that, by construction and independently of $M\geq 0$, the plan of shadow prices associated to debt constraints is zero. Therefore, for any M, money has a zero fundamental value and a bubble under Kuhn-Tucker multipliers. Also, the diversity of individuals beliefs about the uncertainty (probabilities $\rho^h(\xi)$) implies that both agents perceive a finite present value of aggregate wealth.[3]

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[3]Using agent' h Kuhn-Tucker multipliers as deflators, the present value of aggregated wealth at $\xi\in D$, denoted by PV^h_ξ, satisfies,

$$\begin{aligned}
PV^h_\xi &= \sum_{\mu\geq\xi}\frac{\gamma^h_\mu}{\gamma^h_\xi}p_\mu W_\mu=\frac{2}{\rho^h(\xi)}\sum_{\mu\geq\xi}\rho^h(\mu)\beta^{t(\mu)}+\frac{1}{\rho^h(\xi)}\sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\}\}}\rho^h(\mu)\\
&= 2\frac{\beta^{t(\xi)}}{1-\beta}+\sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\},\,t(\mu)\leq t(\xi)+1\}}\frac{\rho^h(\mu)}{\rho^h(\xi)}+\sum_{s=t(\xi)+1}^{+\infty}\left[\frac{1}{2^{s+1}}\left(1-\frac{1}{2^s}\right)+\left(1-\frac{1}{2^{s+1}}\right)\frac{1}{2^s}\right]\\
&= 2\frac{\beta^{t(\xi)}}{1-\beta}+\frac{3}{2}\frac{1}{2^{t(\xi)}}-\frac{1}{3}\frac{1}{4^{t(\xi)}}+\frac{1}{\rho^h(\xi)}\sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\,t(\mu)\leq t(\xi)+1\}}\rho^h(\mu)<+\infty.
\end{aligned}$$

Finally, Assumption B is not satisfied, because aggregated physical endowments were unbounded along the event-tree.[4] □

Appendix A: Duality theory of Individual optimality

Under Assumption A, we will use duality theory to determine necessary conditions for individual optimality. To attempt this objective, we restrict our attention, without loss of generality, to prices $(p,q) \in \mathbb{P} := \{(p,q) \in \mathbb{R}^{L\times D}_+ \times \mathbb{R}^D_+ : (p(\xi),q(\xi)) \in \Delta^{L+1},\ \forall \xi \in D\}$, where, for each $m > 0$, the *simplex* $\Delta^m := \{z = (z_1,\dots,z_m) \in \mathbb{R}^m_+ : \sum_{k=1}^m z_k = 1\}$. Also, remember that the super-gradient of a concave function $f : X \subset \mathbb{R}^L \to \mathbb{R}\cup\{-\infty\}$ at point $x \in X$ is defined as the set of vectors $p \in \mathbb{R}^L$ such that, for all $x' \in X,\ f(\xi,x') - f(\xi,x) \le p(x'-x)$.

For convenience of notations, let $D(\xi) = \{\mu \in D : \mu \ge \xi\}$ be the subtree with root ξ. The set of nodes with date T in $D(\xi)$ is denoted by $D_T(\xi)$. Finally, let $D^T(\xi) = \bigcup_{k=t(\xi)}^T D_k(\xi)$ be the set of successors of ξ with date less or equal than T. When $\xi = \xi_0$ notations above will be shorten to D_T and D^T.

Definition A1. *Given $(p,q) \in \mathbb{P}$ and $y^h = (x^h,z^h) \in B^h(p,q)$, we say that $(\gamma^h(\xi);\xi \in D) \in \mathbb{R}^D_+$ constitutes a family of Kuhn-Tucker multipliers (associated to y^h) if there exists, for each $\xi \in D$, super-gradients $u'(\xi) \in \partial u^h(\xi,x^h(\xi))$ such that,*

(a) For every $\xi \in D$, $\gamma^h(\xi)\, g^h_\xi(y^h(\xi),y^h(\xi^-);p,q) = 0$.

(b) The following Euler conditions hold,

$$\begin{aligned} \gamma^h(\xi)p(\xi) &\ge u'(\xi),\\ \gamma^h(\xi)p(\xi)x^h(\xi) &= u'(\xi)\,x^h(\xi),\\ \gamma^h(\xi)q(\xi) &\ge \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu), \end{aligned}$$

where the last inequality is strict only if the associated debt constraint is binding at ξ.

(c) The following transversality condition holds: $\limsup_{T\to+\infty} \sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z^h(\xi) \le 0$.

Lemma A1. (Finite discounted value of individual endowments)

Fix a plan $(p,q) \in \mathbb{P}$ and $y^h = (x^h,z^h) \in B^h(p,q)$ such that $U^h(x^h) < +\infty$. If Assumption A holds then for any family of Kuhn-Tucker multipliers associated to y^h, $(\gamma^h(\xi);\ \xi \in D)$, we have $\sum_{\xi\in D}\gamma^h(\xi)\ \big(p(\xi)\,w^h(\xi) + q(\xi)e^h(\xi)\big) < +\infty$.

[4] If Assumption B holds, there are $(\delta,\pi) \in \mathbb{R}_{++}\times(0,1)$ such that, for any $\xi \in D^{uu} := \{\mu \in D : \exists \eta \in D; \mu = (\eta^u)^u\}$,

$$\frac{1}{\delta} = \frac{w^h_\xi}{\delta} > \frac{1-\pi}{\beta^{t(\xi)}\rho^h(\xi)}\sum_{\mu>\xi}\rho^h(\mu)\beta^{t(\mu)}W_\mu, \quad \forall h \in H.$$

Thus, for all $(\xi,h) \in D^{uu}\times H$, $\beta^{t(\xi)}\left(\frac{1}{\delta(1-\pi)} + W_\xi\right) > PV^h_\xi$. On the other hand, given $\xi \in D^{uu}$,

$$PV^1_\xi \ge \frac{1}{\rho^1(\xi)}\sum_{\{\mu\ge\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\le t(\xi)+1\}}\rho^1(\mu) = 1 - \frac{1}{2^{t(\xi)+1}}.$$

Therefore, as for any $T \in \mathbb{N}$ there exists $\xi \in D^{uu}$ with $t(\xi) = T$, we conclude that, $\beta^T\left(\frac{1}{\delta(1-\pi)} + 2\right) > 0.5$, for all $T > 0$. A contradiction.

Proof. Let $\mathcal{L}_\xi^h : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the function defined by $\mathcal{L}_\xi^h(y(\xi), y(\xi^-)) = v^h(\xi, y(\xi)) - \gamma^h(\xi)\, g_\xi^h(y(\xi), y(\xi^-); p, q)$, where $y(\xi) = (x(\xi), z(\xi))$ and $v^h(\xi, \cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from Assumption A and Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi \in D^T} \mathcal{L}_\xi^h(0, 0) - \sum_{\xi \in D^T} \mathcal{L}_\xi^h(y^h(\xi), y^h(\xi^-)) \leq - \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi)(0 - z^h(\xi)).$$

Therefore, as for each $\xi \in D$, $\gamma^h(\xi)\, g_\xi^h(y^h(\xi), y^h(\xi^-); p, q) = 0$, we have that, for any $S \in \mathbb{N}$,

$$\begin{aligned} 0 \leq \sum_{\xi \in D^S} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi) e^h(\xi) \right) &\leq \limsup_{T \to +\infty} \sum_{\xi \in D^T} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi) e^h(\xi) \right) \\ &\leq U^h(x^h) + \limsup_{T} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \\ &\leq U^h(x^h) < +\infty, \end{aligned}$$

which concludes the proof. □

Proposition A1. (Necessary conditions for individual optimality)

Fix a plan $(p, q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p, q)$ such that $U^h(x^h) < +\infty$. If Assumption A holds and y^h is an optimal allocation for agent $h \in H$ at prices (p, q), then there exists a family of Kuhn-Tucker multipliers associated to y^h.

Proof. Suppose that $(y^h(\xi))_{\xi \in D}$ is optimal for agent $h \in H$ at prices (p, q). For each $T \in \mathbb{N}$, consider the truncated optimization problem,

$$(P^{h,T}) \qquad \begin{array}{ll} \max & \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ \text{s.t.} & \left\{ \begin{array}{lll} g_\xi^h\left(y(\xi), y(\xi^-);\, p, q\right) & \leq & 0, \quad \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ q(\xi) z(\xi) & \geq & -p(\xi) M, \quad \forall \xi \in D^T \setminus D_T, \\ (x(\xi), z(\eta)) & \geq & 0, \quad \forall (\xi, \eta) \in D^T \times D_T. \end{array} \right. \end{array}$$

It follows that, under Assumption A each truncated problem $P^{h,T}$ has a solution $(y^{h,T}(\xi))_{\xi \in D^T}$.[5] Moreover, the optimality of $(y^h(\xi))_{\xi \in D}$ in the original problem implies that $U^h(x^h)$ is greater than or equal to

[5]In fact, as $(y^h(\xi))_{\xi \in D}$ is optimal and $U^h(x^h) < +\infty$, it follows that there exists a solution for $P^{h,T}$ if and only if there exists a solution for the problem,

$$(\tilde{P}^{h,T}) \qquad \begin{array}{ll} \max & \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ \text{s.t.} & \left\{ \begin{array}{lll} g_\xi^h\left(y(\xi), y(\xi^-);\, p, q\right) & \leq & 0, \quad \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ z(\xi) & \geq & -\frac{p(\xi) M}{q(\xi)}, \quad \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0 \\ z(\xi) & = & 0, \quad \text{if } \left[\xi \in D^{T-1} \text{ and } q(\xi) = 0\right] \text{ or } \xi \in D_T, \\ x(\xi) & \geq & 0, \quad \forall \xi \in D^T. \end{array} \right. \end{array}$$

Indeed, it follows from the existence of an optimal plan which gives finite utility that if $q(\xi) = 0$ for some $\xi \in D$, then $q(\mu) = 0$ for each successor $\mu > \xi$. Now, budget feasibility assures that,

$$z(\xi) \leq \frac{p(\xi) w^h(\xi)}{q(\xi)} + z(\xi^-), \ \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0.$$

As $z(\xi_0^-) = 0$, the set of feasible financial positions is bounded in the problem $(\tilde{P}^{h,T})$. Thus, budget feasible consumption allocations are also bounded and, therefore, the set of admissible strategies is compact. As the objective function is continuous, there is a solution for $(\tilde{P}^{h,T})$.

$\sum_{\xi\in D^T} u^h(\xi, x^{h,T}(\xi))$. In fact, the plan $(\tilde{y}_\xi)_{\xi\in D}$ defined by $\tilde{y}_\xi = y_\xi^{h,T}$, for each $\xi \in D^T$, and by $\tilde{y}_\xi = 0$ otherwise, is budget feasible in the original economy and, therefore, the allocation $(y^{h,T}(\xi))_{\xi\in D^T}$ cannot improve the utility level of agent h.

Define $v^h(\xi,\cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{in other case.} \end{cases}$$

where $y(\xi) = (x(\xi), z(\xi))$. Given a multiplier $\gamma \in \mathbb{R}$, let $\mathcal{L}_\xi^h(\cdot, \gamma; p, q) : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the Lagrangian at node ξ, i.e.,

$$\mathcal{L}_\xi^h(y(\xi), y(\xi^-), \gamma;\, p, q) = v^h(\xi, y(\xi)) - \gamma\, g_\xi^h(y(\xi), y(\xi^-); p, q).$$

It follows from Rockafellar (1997, Theorem 28.3) that there exist non-negative multipliers $(\gamma^{h,T}(\xi))_{\xi\in D^T}$ such that the following saddle point property

$$\sum_{\xi\in D^T} \mathcal{L}_\xi^h(y(\xi), y(\xi^-), \gamma^{h,T}(\xi);\, p, q) \leq \sum_{\xi\in D^T} \mathcal{L}_\xi^h(y^{h,T}(\xi), y^{h,T}(\xi^-), \gamma^{h,T}(\xi);\, p, q), \tag{1}$$

is satisfied, for each plan $(y(\xi))_{\xi\in D^T} = (x(\xi), z(\xi))_{\xi\in D^T}$ for which

$$\begin{aligned} (x(\xi), z(\eta)) \quad &\geq \quad 0, & \forall (\xi, \eta) \in D^T \times D_T, \\ q(\xi)z(\xi) \quad &\geq \quad -p(\xi)M, & \forall \xi \in D^T \setminus D_T. \end{aligned}$$

Moreover, at each node $\xi \in D^T$, multipliers satisfy $\gamma^{h,T}(\xi)\, g_\xi^h(y^{h,T}(\xi), y^{h,T}(\xi^-); p, q) = 0$.

Analogous arguments to those made in Claims A1-A3 in Araujo, Páscoa and Torres-Martínez (2007) implies that,

Claim. *Under Assumption A, the following conditions hold:*
(i) For each $t < T$,

$$0 \leq \sum_{\xi\in D^t} \gamma^{h,T}(\xi) \left(p(\xi)w^h(\xi) + q(\xi)e^h(\xi)\right) \leq U^h(x^h).$$

(ii) For each $0 < t < T$,

$$\sum_{\xi\in D_t} \gamma^{h,T}(\xi)q(\xi)z^h(\xi^-) \leq \sum_{\xi\in D\setminus D^{t-1}} u^h(\xi, x^h(\xi)).$$

(iii) For each $\xi \in D^{T-1}$ and for any $y(\xi) = (x(\xi), z(\xi))$, with $x(\xi) \geq 0$ and $q(\xi)z(\xi) \geq -p(\xi)M$,

$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \leq \left(\gamma^{h,T}(\xi)p(\xi)\,;\; \gamma^{h,T}(\xi)q(\xi) - \sum_{\mu\in\xi^+} \gamma^{h,T}(\mu)q(\mu)\right) \cdot (y(\xi) - y^h(\xi))$$
$$+ \sum_{\eta\in D\setminus D^T} u^h(\eta, x^h(\eta)).$$

Now, at each $\xi \in D$, $\underline{w}^h(\xi) := \min_{l\in L} w^h(\xi, l) > 0$. Also, as a consequence of monotonicity of $u^h(\xi)$, $||p(\xi)||_\Sigma > 0$. Thus, item (i) above guarantees that, for each $\xi \in D$,

$$0 \leq \gamma^{h,T}(\xi) \leq \frac{U^h(x^h)}{\underline{w}^h(\xi)\, ||p(\xi)||_\Sigma}, \quad \forall T > t(\xi).$$

Therefore, the sequence $(\gamma^{h,T}(\xi))_{T\geq t(\xi)}$ is bounded, node by node. As the event-tree is countable, there is a common subsequence $(T_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ and non-negative multipliers $(\gamma^h(\xi))_{\xi\in D}$ such that, for each $\xi\in D$, $\gamma^{h,T_k}(\xi)\to_{k\to+\infty}\gamma^h(\xi)$, and

$$\gamma^h(\xi)g^h_\xi(p,q,y^h(\xi),y^h(\xi^-)) \quad = \quad 0\,; \tag{2}$$

$$\limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi^-) \quad \leq \quad 0\,, \tag{3}$$

where equation (2) follows from the strictly monotonicity of $u^h(\xi)$, and equation (3) is a consequence of item (ii) (taking the limit as T goes to infinity and, afterwards, the limit in t).

Moreover, using item (iii), and taking the limit as T goes to infinity, we obtain that, for each $y(\xi)=(x(\xi),z(\xi))$, with $x(\xi)\geq 0$ and $q(\xi)z(\xi)\geq -p(\xi)M$,

$$u^h(\xi,x(\xi))-u^h(\xi,x^h(\xi))\leq\left(\gamma^h(\xi)p(\xi)\,;\;\gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)\cdot(y(\xi)-y^h(\xi)).$$

Let $\mathcal{F}^h(\xi,p,q)=\{(x,z)\in\mathbb{R}^L\times\mathbb{R}:\,x\geq 0\ \wedge\ q(\xi)z\geq -p(\xi)M\}$.

It follows that $\left(\gamma^h(\xi)p(\xi)\,;\,\gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)$ belongs to the super-differential set of the function $v^h(\xi,\cdot)+\delta(\cdot,\mathcal{F}^h(\xi,p,q))$ at point $y^h(\xi)$, where $\delta(y,\mathcal{F}^h(\xi,p,q))=0$, when $y\in\mathcal{F}^h(\xi,p,q)$ and $\delta(y,\mathcal{F}^h(\xi,p,q))=-\infty$, otherwise. Notice that, for each $y\in\mathcal{F}^h(\xi,p,q)$, $\kappa\in\partial\delta(y,\mathcal{F}^h(\xi,p,q))\Leftrightarrow 0\leq k(y'-y),\ \forall y'\in\mathcal{F}^h(\xi,p,q)$.

Now, by Theorem 23.8 in Rockafellar (1997), for all $y\in\mathcal{F}^h(\xi,p,q)$, if $v'(\xi)$ belongs to $\partial\left[v^h(\xi,y)+\delta(y,\mathcal{F}^h(\xi,p,q))\right]$ then there exists $\tilde v'(\xi)\in\partial v^h(\xi,y)$ such that both $v'(\xi)\geq\tilde v'(\xi)$ and $(v'(\xi)-\tilde v'(\xi))\cdot(x,q(\xi)z+p(\xi)M)=0$, where $y=(x,z)$. Therefore, it follows that there exists, for each $\xi\in D$, a super-gradient $\tilde v'(\xi)\in\partial v^h(\xi,y^h(\xi))$ such that,

$$\left(\gamma^h(\xi)p(\xi)\,;\;\gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)-\tilde v'(\xi)\quad\geq\quad 0,$$

$$\left[\left(\gamma^h(\xi)p(\xi)\,;\;\gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)-\tilde v'(\xi)\right]\cdot(x^h(\xi),q(\xi)z^h(\xi)+p(\xi)M)\quad=\quad 0.$$

As $\tilde v'(\xi)\in\partial v^h(\xi,y^h(\xi))$ if and only if there is $u'(\xi)\in\partial u^h(\xi,x^h(\xi))$ such that $\tilde v'(\xi)=(u'(\xi),0)$, it follows from last inequalities that Euler conditions hold.

On the other side, item (i) in claim above guarantees that, $\sum_{\xi\in D}\gamma^h(\xi)(p(\xi)w^h(\xi)+q(\xi)e^h(\xi))<+\infty$ and, therefore, equations (2) and (3) assure that,

$$\begin{aligned}\limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi) \quad &\leq \quad \limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)\left(p(\xi)w^h(\xi)+q(\xi)e^h(\xi)+q(\xi)z^h(\xi^-)\right)\\ &\leq \quad \limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi^-)\ \leq\ 0,\end{aligned}$$

which implies that transversality condition holds. □

Note that we could prove, alternatively, the existence of a state price deflator that satisfies *financial* Euler equation using, as Santos and Woodford (1997), non-arbitrage conditions. However, to attempt our objectives we need to assure that Kuhn-Tucker deflators exist, in the sense of Definition A1, and also that the discounted value of endowments, using these deflators, is finite.

On the other hand, as under Kuhn-Tucker multipliers the deflated value of individual endowments is finite, our transversality condition is equivalent to the requirement imposed by Magill and Quinzii (1996), provided that either short sales were avoided or individual endowments were uniformly bounded away from zero.

Corollary.

Fix $(p,q) \in \mathbb{P}$. Under Assumption A, given $h \in H$ suppose that either $M = 0$ or there exists $\underline{w} \in \mathbb{R}^L_{++}$ such that, at any $\xi \in D$, $w^h(\xi) \geq \underline{w}$. If y^h is an optimal allocation for agent h at prices (p,q), then for any plan of Kuhn-Tucker multipliers associated to y^h, $(\gamma^h(\xi))_{\xi \in D}$, we have,

$$\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0.$$

Proof. Let $(\gamma^h(\xi))_{\xi \in D}$ be a plan of Kuhn-Tucker multipliers associated to y^h. We know that the transversality condition of Definition A1 holds. On the other hand, it follows directly from the debt constraint that,

$$\sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \geq - \sum_{\xi \in D_T} \gamma^h(\xi) p(\xi) M \geq - \left(\max_{l \in L} M_l \right) \sum_{\xi \in D_T} \gamma^h(\xi) \|p(\xi)\|_\Sigma.$$

Therefore, when $M = 0$ we obtain the result. Alternatively, assume that for any $\xi \in D$, $w^h(\xi) \geq \underline{w}$. As by Lemma A1 the sum $\sum_{\xi \in D} \gamma^h(\xi) p(\xi) w^h(\xi)$ is well defined and finite we have that

$$\sum_{\xi \in D} \gamma^h(\xi) \|p(\xi)\|_\Sigma < +\infty.$$

Thus, $\liminf_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \geq 0$ which implies, using the transversality condition of Definition A1, that $\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0$. □

We end this Appendix with a result that determines sufficient requirements to assure that a plan of consumption and portfolio allocations is individually optimal. Note that the result below will assure that, when either short-sales were avoided, i.e. $M = 0$, or individual endowments were uniformly bounded away from zero, a budget feasible plan is individually optimal *if and only if* there exists a family of Kuhn-Tucker multipliers associated to it.

Proposition A2. (Sufficient conditions for individual optimality)

Fix a plan $(p,q) \in \mathbb{P}$. Under Assumption A, suppose that given $y^h = (x^h, z^h) \in B^h(p,q)$, there exists a family of Kuhn-Tucker multipliers $(\gamma^h(\xi); \xi \in D)$ associated to y^h. Then, if

$$\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) p(\xi) M \quad = \quad 0,$$

then y^h is an optimal allocation for agent h at prices (p,q).

Proof. Note that, under the conditions above $\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0$. On the other hand, it follows from Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi \in D^T} \mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma^h(\xi); p, q) - \sum_{\xi \in D^T} \mathcal{L}^h_\xi(y^h(\xi), y^h(\xi^-), \gamma^h_\xi; p, q) \leq - \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi)(z(\xi) - z^h(\xi)).$$

Moreover, as at each node $\xi \in D$ we have that $\gamma^h(\xi) g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$, each budget feasible allocation $y = ((x(\xi), z(\xi)); \xi \in D)$ must satisfy

$$\sum_{\xi \in D^T} u^h(\xi, x(\xi)) - \sum_{\xi \in D^T} u^h(\xi, x^h(\xi)) \leq - \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi)(z(\xi) - z^h(\xi)).$$

Now, as the sequence $\left(\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z^h(\xi)\right)_{T\in\mathbb{N}}$ converges, it is bounded. Thus,

$$\begin{aligned}\limsup_{T\to+\infty}\left(-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi))\right) &\leq \limsup_{T\to+\infty}\left(-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z(\xi)\right)\\ &\leq \lim_{T\to+\infty}\sum_{\xi\in D_T}\gamma^h(\xi)p(\xi)M=0.\end{aligned}$$

Therefore,

$$U^h(x)=\limsup_{T\to+\infty}\sum_{\xi\in D^T}u^h(\xi,x(\xi))\leq U^h(x^h),$$

which guarantees that the allocation $(x^h(\xi),z^h(\xi))_{\xi\in D}$ is optimal. □

Appendix B: On the fundamental value of money

In the frictionless theory developed by Santos and Woodford (1997), in which debt constraints are non saturated, two (equivalent) definitions of the fundamental value of money make economic sense. The fundamental value is either (1) equal to the discounted value of future deliveries that an agent will receive for one unit of money that he buys and keeps forever; (2) equal to the discounted value of rental services, that coincides with deliveries, given the absence of any friction associated to debt constraint.

These concepts do not coincide when frictions are allowed. Thus, we adopt the second definition, that internalize the role that money has: it allows for inter-temporal transfers, although its deliveries are zero.

Proposition B1. (Non-existence of negative bubbles)

Under Assumption A, given an equilibrium $\left[(p,q);((x^h,z^h);h\in H)\right]$, *at each node* $\xi\in D$, $q(\xi)\geq F(\xi,q,\gamma^h(\xi))$, *where* $(\gamma^h(\xi);\xi\in D)$ *denotes the agent's h plan of Kuhn-Tucker multipliers and*

$$F(\xi,q,\gamma^h(\xi)):=\frac{1}{\gamma^h(\xi)}\sum_{\mu\in D(\xi)}\left(\gamma^h(\mu)q(\mu)-\sum_{\nu\in\mu^+}\gamma^h(\nu)q(\nu)\right),$$

is the fundamental value of money at $\xi\in D$.

Proof. By Proposition A1, there are, for each agent $h\in H$, non-negative shadow prices $(\eta^h(\xi);\xi\in D)$, satisfying for each $\xi\in D$,

$$\begin{aligned}0 &= \eta^h(\xi)\left(q(\xi)z^h(\xi)+p(\xi)M\right);\\ \gamma^h(\xi)q(\xi) &= \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)+\eta^h(\xi)q(\xi).\end{aligned}$$

Therefore,

$$\gamma^h(\xi)q(\xi)=\sum_{\mu\geq\xi}\eta^h(\mu)q(\mu)+\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\gamma^h(\mu)q(\mu).$$

As multipliers and monetary prices are non-negative, the infinite sum in the right hand side of equation above is well defined, because its partial sums are increasing and bounded by $\gamma^h(\xi)q(\xi)$. This also implies that the limit of the (discounted) asset price exists. □

Note that the rental services that one unit of money gives at $\mu\in D$ are equal to $q(\mu)-\sum_{\nu\in\mu^+}\frac{\gamma^h(\nu)}{\gamma^h(\mu)}q(\mu)$. Thus, the *fundamental value* of money at a node ξ, as was defined in Proposition B1, coincides with the discounted value of (unitary) future rental services.

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