TEXTO PARA DISCUSSÃO

No. 545

With additional enforcement mechanisms, does collateral avoid Ponzi Schemes?

Thiago Revil
Juan Pablo Torres-Martínez

DEPARTAMENTO DE ECONOMIA
www.econ.puc-rio.br

WITH ADDITIONAL ENFORCEMENT MECHANISMS, DOES COLLATERAL AVOID PONZI SCHEMES ?

THIAGO REVIL AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. In infinite horizon incomplete market economies, when collateral repossession is the only mechanism enforcing borrowers not to entirely default on their promises, Ponzi schemes are avoided and equilibrium exists independently of the choice of collateral bundles.

In these economies, we add default enforcement mechanisms that are *effective*, i.e. induce payments besides the value of collateral guarantees. We prove that, independently of prices, the individual's problem does not have a physically feasible solution when collateral guarantees are not large enough relative to the effectiveness of the additional enforcement mechanisms. We also emphasize that this result does not depend on specific types of such mechanisms, as long lenders face payments besides the value of collateral guarantees as a percentage of the remaining debt.

Keywords. Effective default enforcements, Collateral repossession, Individual's optimality.
JEL classification: D50; D52.

1. Introduction

In modern financial markets, collateral guarantees play an important role in enforcing borrowers not to entirely default on their financial promises. These guarantees are used in several credit operations, from corporate bonds to Collateralized Mortgages Obligations,[1] allowing markets to reduce credit risk and increase portfolio diversification. However, to protect investors from the excess of losses induced by large negative shocks in the value of collateral guarantees, financial markets may create and implement additional enforcement mechanisms against default. In this paper, we focus on the theoretical effects of this policy.

In general equilibrium models, the collateralization of financial contracts is mostly addressed when the only default enforcement mechanism is the seizure of the associated collateral guarantees.[2] In infinite horizon models with incomplete markets, and without exogenous debt constraints or transversality conditions, Araujo, Páscoa and Torres-Martínez (2002) proved the existence of equilibrium independently of the choice of physical collateral guarantees. Essentially, when collateral repossession is the only default enforcement mechanism, non-arbitrage conditions ensure that the

Date: November, 2007 (revised version).

We are indebted with Filipe Martins-da-Rocha for suggestions on previous versions of this paper. We thank Marcelo Abreu, Helen Harris, Carlos Hervés-Beloso and Emma Moreno-Garcia for useful comments. T. Revil acknowledges support from CNPq. J.P.Torres-Martínez receives a CNPq research grant through project 307554/2004-0.

[1] That is, derivative assets secured by pools of individual mortgages, each of which is backed mostly by real estates.

[2] For a seminal two-period general equilibrium model of collateralized loans, see Geanakoplos and Zame (2002), and for an extension to more complex securitization structures see Steinert and Torres-Martínez (2007).

price of the joint operation of taking a loan and constituting the respective collateral requirements is always non-negative, eliminating Ponzi schemes. Also, in such a context, as the existence of collateral requirements rationalize debt constraints, computational methods can be used to approximate equilibrium allocations for any choice of collateral bundles (see Kubler and Schmedders (2003)).

In the economy studied by Araujo, Páscoa and Torres-Martínez (2002), we add default enforcement mechanisms that are *effective*, i.e. enforce payments besides the value of the collateral guarantees. In this context, if these additional enforcement mechanisms are persistently effective, we prove our main result: independently of prices, the individual's problem does not have a physically feasible solution when collateral requirements are not large enough relative to the effectiveness of such mechanisms. For this result, we provide upper bounds for collateral bundles in terms of the primitives of the economy and we only need additional mechanisms to become persistently effective in at least one path of uncertainty, even if the probability of such event is asymptotically zero.

We represent additional default enforcement mechanisms by their effectiveness on enforcing payments besides the value of the collateral requirements. With this approach, we can focus on the causes generating the non-existence of a solution for the individual's problem. Also, we can include several types of mechanisms in our analysis, provided that their effectiveness can be faced by lenders either as a percentage of the remanning debt or as the imposition of a pecuniary default penalty.

Previously, Páscoa and Seghir (2006) have shown that the individual's problem may not have a physically feasible solution when the only enforcement mechanism besides collateral repossession is given by linear utility penalties for default. They provide examples of economies in which those penalties are harsh, implying in loan values greater than that of the associated collateral requirements, which, then, lead to the nonexistence of an optimal plan compatible with the available physical resources.[3] However, it follows from our main result that the nonexistence of a physically feasible solution for the individual's problem does not depend on specific types of additional enforcement mechanisms. Also, as we show, it is not necessary to ensure that borrowers honor a high percentage of the original promises, it is sufficient to have collateral requirements that are not large enough in a context of persistently effective additional enforcement mechanisms.

In the infinite horizon economies with collateral repossession here studied, the mere presence of additional effective enforcement mechanisms does not necessarily eliminate the existence of equilibria. In fact, if non-arbitrage conditions ensure that the difference between the value of the collateral requirements and that of the associated loan is always non-negative, then, arguments analogous to those made by Araujo, Páscoa and Torres-Martínez (2002) imply equilibrium existence. However, in such a context, we claim that the choice of collateral guarantees becomes relevant to ensure the non-negativity of the difference above.

The remainder of the paper is organized as follows: Section 2 presents an infinite horizon economy with assets subject to default and with effective enforcement mechanisms in addition to collateral repossession. In Section 3 we show our main result and some extensions are discussed in Section 4.

[3]Since the additional enforcement mechanism studied by these authors may become effective only when these penalties are harsh, they impose upper bounds on utility penalties to ensure the existence of equilibrium (see Theorem 4.1 in Páscoa and Seghir (2006)).

2. Model

Consider a discrete time, infinite horizon economy with uncertainty and symmetric information. Let S be the set of states of nature and $\mathbb{F}_t$ the information available at period $t \in T := \mathbb{N} \cup \{0\}$. $\mathbb{F}_t$ is a partition of S, and if $t' > t$, make $\mathbb{F}_{t'}$ finer than $\mathbb{F}_t$. Summarizing the uncertainty structure, define an event-tree as $D = \{(t,\sigma) \in T \times 2^S : t \in T, \sigma \in \mathbb{F}_t\}$, where a pair $\xi := (t,\sigma) \in D$ is called a node and $t(\xi) := t$ is the associated period of time. For simplicity, at $t = 0$ there is no information, $\mathbb{F}_0 := S$, and there is only one node, ξ_0.

A node $\xi' = (t', \psi')$ is a successor of $\xi = (t, \psi)$, denoted by $\xi' \geq \xi$, if $t' \geq t$ and $\psi' \subseteq \psi$. given $\xi \in D$, the set of its successors is given by the subtree $D(\xi) := \{\mu \in D : \mu \geq \xi\}$. Also, for each $\xi \neq \xi_0$, since $\mathbb{F}_{t(\xi)}$ is finer than $\mathbb{F}_{t(\xi)-1}$, there is only one predecessor ξ^-. We define ξ' as an immediate successor of ξ when it is in the set $\xi^+ := \{\xi' \in D : \xi' \geq \xi, t(\xi') = t(\xi) + 1\}$.

Given $k \in \mathbb{N} \cup \{+\infty\}$, we call path of uncertainty any set of nodes $(\mu_n; n \in \mathbb{N}, n \leq k) \subset D$ in which every μ_{n+1} is an immediate successor of μ_n, for each $n < k$. A set $B \subset D$ does not have finite paths when for any $k \in \mathbb{N}$ and for each path of uncertainty $(\mu_n; n \in \mathbb{N}, n \leq k) \subset B$, there exists $\eta \in B$ such that $\eta \in \mu_k^+$.

At each node ξ in the event-tree D there is a non-empty and finite set of commodities, L. These commodities may be traded in a competitive market at unitary prices $p_\xi = (p_{(\xi,l)})_{l \in L} \in \mathbb{R}_+^L$ by a non-empty set of consumers and, at the same time, may depreciate from a node to its successors. Along the event-tree, this depreciation follows a technology represented by a family of matrices with non-negative entries, $(Y_\xi)_{\xi \in D}$, where $Y_\xi := ((Y_\xi)_{l,l'})_{(l,l') \in L \times L}$. For each $(l,l') \in L \times L$, $(Y_\xi)_{l,l'}$ is the amount of commodity l obtained at ξ if one unit of commodity l' is consumed at ξ^-. Also, let $W_\xi \in \mathbb{R}_+^L$ be the aggregate physical resources up to node ξ, while $W = (W_\xi)_{\xi \in D}$ is the plan of such resources.

There is a finite set of real assets $J(\xi)$ at each node $\xi \in D$. Each $j \in J(\xi)$ is short-lived, has promises $A_{(\mu,j)} \in \mathbb{R}_{++}^L \cup \{0\}$ at $\mu \in \xi^+$, and is traded in competitive markets by a unitary price $q_{(\xi,j)} \in \mathbb{R}_+$. Note that, the non-triviality of financial promises implies that its market value take into account all the commodities prices. This assumption may be intuitively understood as an indexation for asset payments using a price index of a referential bundle that may vary with the uncertainty of the economy. Thus, independently of prices, when at least a percentage of original promises is honored by borrowers, lenders maintain a minimal purchase power for every commodity.

Since assets are subject to credit risk, borrowers are burdened to constitute physical collateral guarantees in order to limit lenders' losses. Particularly, for every unit of an asset $j \in J(\xi)$ sold, borrowers must establish—and may consume—a bundle $C_{(\xi,j)} \in \mathbb{R}_+^L \setminus \{0\}$ that is seized by the market in case of default. For the sake of notation, let $J(D) := \{(\xi, j) \in D \times \cup_{\mu \in D} J(\mu) : j \in J(\xi)\}$ and $J^+(D) := \{(\mu, j) \in D \times \cup_{\eta \in D} J(\eta) : (\mu^-, j) \in J(D)\}$.

Furthermore, additional default enforcement mechanisms may exist. We let financial markets recover, at each $(\mu, j) \in J^+(D)$, amounts of payments $F_{(\mu,j)}(p_\mu)$ that may be higher, in case of default, than the value of depreciated collateral guarantees. In these payments, we allow generality by assuming that additional enforcement mechanisms may induce borrowers to pay: a fixed percentage of the remaining debt, $\lambda_{(\mu,j)} \in [0,1]$, and/or the market value of a given bundle of resources,

$y_{(\mu,j)} \in \mathbb{R}^L_+$. These exogenous variables may be intuitively interpreted as a probability that the judicial system imposes the entire payment of the remaining debt, and a real pecuniary default penalty. More formally, for every unit of asset $j \in J(\xi)$, each borrower pays at each $\mu \in \xi^+$ an amount

$$F_{(\mu,j)}(p_\mu) := \min\{p_\mu A_{(\mu,j)}, p_\mu Y_\mu C_{(\xi,j)}\} + \lambda_{(\mu,j)}\, [p_\mu A_{(\mu,j)} - p_\mu Y_\mu C_{(\xi,j)}]^+ \\ + \min\{p_\mu y_{(\mu,j)}, (1-\lambda_{(\mu,j)})\, [p_\mu A_{(\mu,j)} - p_\mu Y_\mu C_{(\xi,j)}]^+\},$$

where $(\lambda_{(\mu,j)}, y_{(\mu,j)}) \in [0,1]\times(\mathbb{R}^L_{++}\cup\{0\})$ is the *effectiveness* of additional enforcement mechanisms on asset j at node μ, and, for any $z \in \mathbb{R}$, $[z]^+ := \max\{z,0\}$.

Regarding the additional enforcement mechanisms above, we do not intend to explicitly model how the market imposes on borrowers additional payments besides the value of collateral guarantees. We represent such mechanisms by their effectiveness. This approach allows us to include in our analysis economic (i.e. those induced by legal contracts) and non-economic (e.g. moral sanctions, loss of reputations) default enforcement mechanisms, provided that these mechanisms may be summarized by a vector of effectiveness. Most importantly, with this approach, it is possible to focus on the consequences of the effectiveness of such mechanisms on the individual's decision.

DEFINITION. *Given $(\mu,j) \in J^+(D)$, additional enforcement mechanisms are effective on asset j at node μ when both $(\lambda_{(\mu,j)}, y_{(\mu,j)})$ and $A_{(\mu,j)}$ are non-zero vectors. Additional enforcement mechanisms are persistently effective in a path of uncertainty Θ, if for any $\mu \in \Theta$, there is $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ.*

For any path of uncertainty $\Theta := (\mu_n; n \in \mathbb{N})$ in which additional enforcement mechanisms are persistently effective, define $\mathrm{Eff}(\Theta) \subset D(\mu_1)$ as the maximal connected set containing Θ and having, at each $\mu \in \mathrm{Eff}(\Theta)$, at least one $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ.[4] Note that, given $(\mu,j) \in J^+(D)$, those definitions above not only depend on the parameters $(\lambda_{(\mu,j)}, y_{(\mu,j)})$, but also on the non-triviality of the original promises. Thus, effective additional enforcement mechanisms means that, in the case of default, a strictly positive amount of resources is seized besides the depreciated collateral value.

In contrast to any equilibrium model, we focus in the nonexistence of a physically feasible solution for the individual's problem and its relationship with the existence of opportunities to implement Ponzi schemes. For these reasons, it is sufficient to study a decision model where there is an infinitely lived agent, namely i, who perfectly foresees both market prices and the effectiveness of additional enforcement mechanisms.

Agent i has physical endowments $(w^i_\xi)_{\xi\in D} \in \mathbb{R}^{D\times L}_+$ and preferences represented by a utility function $U^i : \mathbb{R}^{D\times L}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. As commodities may be durable, we denote by W^i_ξ the cumulated endowments of agent i up to node ξ, which are recursively defined by: $W^i_\xi = w^i_\xi + Y_\xi W^i_{\xi^-}$, when $\xi > \xi_0$, and $W^i_{\xi_0} = w^i_{\xi_0}$, otherwise. Let $x_\xi \in \mathbb{R}^L_+$ be a bundle of autonomous consumption

[4] A set $B \subset D$ is connected when, for each pair $(\xi,\mu) \in B\times B$, such that $\mu \geq \xi$, the only path of uncertainty connecting ξ to μ is contained in B. Given $\xi \in D$, a set $B \subset D(\xi)$ is maximal relative to a property A when there is no other subset of $D(\xi)$ containing itself and satisfying A.

at node ξ (i.e. non-collateralized commodities). Also, let $\theta_{(\xi,j)}$ and $\varphi_{(\xi,j)}$ be quantities of asset $j\in J(\xi)$ purchased and sold at the same node. Given $(p,q)\in\Pi:=\mathbb{R}_+^{D\times L}\times\mathbb{R}_+^{J(D)}$, a plan

$$(x,\theta,\varphi):=\big((x_\xi,\theta_{(\xi,j)},\varphi_{(\xi,j)});\xi\in D,\,j\in J(\xi)\big)\in\mathbb{E}:=\mathbb{R}_+^{D\times L}\times\mathbb{R}_+^{J(D)}\times\mathbb{R}_+^{J(D)}$$

is *budget feasible* for agent i at prices (p,q) when

$$p_{\xi_0}(x_{\xi_0}-w^i_{\xi_0})+p_{\xi_0}\sum_{j\in J(\xi_0)}C_{(\xi_0,j)}\varphi_{(\xi_0,j)}+\sum_{j\in J(\xi_0)}q_{(\xi_0,j)}(\theta_{(\xi_0,j)}-\varphi_{(\xi_0,j)})\ \le\ 0, \tag{1}$$

$$\begin{aligned}p_\xi(x_\xi-w^i_\xi)+p_\xi\sum_{j\in J(\xi)}C_{(\xi,j)}\varphi_{(\xi,j)}+\sum_{j\in J(\xi)}q_{(\xi,j)}(\theta_{(\xi,j)}-\varphi_{(\xi,j)})\\ \le p_\xi Y_\xi x_{\xi^-}+\sum_{j\in J(\xi^-)}\left(p_\xi Y_\xi C_{(\xi^-,j)}\varphi_{(\xi^-,j)}+F_{(\xi,j)}(p_\xi)(\theta_{(\xi^-,j)}-\varphi_{(\xi^-,j)})\right),\ \ \forall\xi>\xi_0.\end{aligned} \tag{2}$$

Also, $(x,\theta,\varphi)\in\mathbb{E}$ is *physically feasible* if $x_\xi+\sum_{j\in J(\xi)}C_{(\xi,j)}\varphi_{(\xi,j)}\le W_\xi$, for any $\xi\in D$. Finally, given $(p,q)\in\Pi$, the objective of agent i is to maximize the utility of his consumption, $U^i((x^i_\xi+\sum_{j\in J(\xi)}C_{(\xi,j)}\varphi^i_{(\xi,j)})_{\xi\in D})$, choosing a budget feasible plan $(x^i,\theta^i,\varphi^i)\in\mathbb{E}$.

3. Enforcement mechanisms and the size of collateral bundles

In this section, we prove our main result: in contrast to the polar case studied by Araujo, Páscoa and Torres-Martínez (2002), the market choice of collateral bundles becomes relevant when there are *persistently effective* additional enforcement mechanisms besides collateral repossession. To achieve our objective, we impose the following hypotheses.

Assumption A1. *For any $\xi\in D$, $W^i_\xi\gg 0$.*

Assumption A2. *Given $z=(z_\xi)\in\mathbb{R}_+^{L\times D}$, define $U^i(z)=\sum_{\xi\in D}u^i_\xi(z_\xi)$, where for any $\xi\in D$, the function $u^i_\xi:\mathbb{R}^L_+\to\mathbb{R}_+$ is concave, continuous and strictly increasing. Also, $U^i(W)$ is finite.*[5]

Given $\eta\in D$, let $\Omega(\eta)$ be the set of assets $j\in J(\eta)$ on which additional enforcement mechanisms are effective at some node $\mu\in\eta^+$. Note that, given a path of uncertainty Θ, in which additional enforcement mechanisms are persistently effective, if $\mathrm{Eff}(\Theta)$ does not have finite paths, then $\Omega(\eta)\neq\emptyset,\ \forall\,\eta\in\mathrm{Eff}(\Theta)$.

Theorem. *Under Assumptions A1-A2, suppose that additional enforcement mechanisms are persistently effective in a path of uncertainty Θ and that $\mathrm{Eff}(\Theta)$ does not have finite paths. Independently of the prices $(p,q)\in\Pi$, there are strictly positive upper bounds, $(\Psi_\eta)_{\eta\in\mathrm{Eff}(\Theta)}$, such that if*

$$\min_{j\in\Omega(\eta)}\|C_{(\eta,j)}\|_\Sigma<\Psi_\eta,\qquad\forall\,\eta\in\mathrm{Eff}(\Theta),$$

[5] Note that, as utilities are finite when evaluated in aggregate physical resources, the non-negativity of functions $u^i_\xi(\cdot)$ implies that, in any physical feasible allocation, agent's i utility is finite. Also, the concavity of the functions $(u^i_\xi)_{\xi\in D}$ implies that U^i is concave. Thus, for any $\sigma>1$, $U^i(\sigma W)$ is also finite. In fact, $U^i(0.5W)<+\infty$ and, therefore, by concavity, $U^i(W)\ge\tau U^i(0.5W)+(1-\tau)U^i(\sigma W)$, where $\tau=\frac{2\sigma-2}{2\sigma-1}\in(0,1)$, which implies that $U^i(\sigma W)<+\infty$.

then agent i's problem does not have a physically feasible solution.

PROOF. To shorten the notation, given $z=(z_1,\dots,z_m)\in\mathbb{R}^m_+$, let $\|z\|_\Sigma := \sum_{s=1}^m z_s$ and $\|z\|_{\max} := \max_{1\le s\le m}\ z_s$. Fix $\sigma>1$. Given $\eta\in \mathrm{Eff}(\Theta)$, define

$$\overline{\pi}_\eta := \frac{U^i(W)}{\min_{l\in L} W^i_{(\eta,l)}}, \qquad \text{and} \qquad \underline{\pi}_\eta := \frac{u^i_\eta(\sigma W_\eta)-u^i_\eta(W_\eta)}{\sigma\|W_\eta\|_{\max}}.$$

Thus, for each $\eta\in\mathrm{Eff}(\Theta)$,

$$\Upsilon_\eta := \min_{j\in\Omega(\eta)}\sum_{\mu\in\eta^+}\min\left\{\lambda_{(\mu,j)}\underline{\pi}_\mu \min_{l\in L} A_{(\mu,j,l)} + \underline{\pi}_\mu \min_{l\in L} y_{(\mu,j,l)}\ ,\ \underline{\pi}_\mu \min_{l\in L} A_{(\mu,j,l)}\right\}$$

is strictly positive, where $A_{(\mu,j,l)}$ (resp. $y_{(\mu,j,l)}$) denotes the l-th coordinate of $A_{(\mu,j)}$ (resp. $y_{(\mu,j)}$). Thus, suppose that, at each $\eta\in\mathrm{Eff}(\Theta)$,

$$\min_{j\in\Omega(\eta)}\|C_{(\eta,j)}\|_\Sigma < \Psi_\eta := \frac{\Upsilon_\eta}{\overline{\pi}_\eta}.$$

Assume that, for some $(p,q)\in\Pi$, there is an optimal budget and physically-feasible solution $(x^i,\theta^i,\varphi^i)\in\mathbb{E}$ for agent i's problem. It follows from Lemma 2 (see Appendix) that there are, for every $\eta\in D$, multipliers $\gamma^i_\eta\in\mathbb{R}_{++}$ and non-pecuniary returns (super-gradients) $v^i_\eta\in \partial u^i_\eta\left(x^i_\eta+\sum_{j\in J(\eta)}C_{(\eta,j)}\varphi^i_{(\eta,j)}\right)$ such that,[6] for each $j\in J(\eta)$,

$$\gamma^i_\eta p_\eta \geq v^i_\eta + \sum_{\mu\in\eta^+}\gamma^i_\mu p_\mu Y_\mu, \tag{3}$$

$$\gamma^i_\eta q_{(\eta,j)} \geq \sum_{\mu\in\eta^+}\gamma^i_\mu F_{(\mu,j)}(p_\mu). \tag{4}$$

Also, the family of multipliers $(\gamma^i_\eta)_{\eta\in D}$ can always be constructed to satisfy (see Lemma 2)

$$\gamma^i_\eta p_\eta W^i_\eta \le \sum_{\eta\in D} u^i_\eta\left(x^i_\eta + \sum_{j\in J(\eta)} C_{(\eta,j)}\varphi^i_{(\eta,j)}\right) \le \sum_{\eta\in D} u^i_\eta(W_\eta), \tag{5}$$

where the last inequality follows from Assumption A2 jointly with the physical feasibility of agent i's consumption. Moreover, it is possible to find lower and upper bounds for $\gamma^i_\eta p_\eta$ at each $\eta\in D$. Assumption A1 and equation (5) ensure that $\gamma^i_\eta\|p_\eta\|_\Sigma\le\overline{\pi}_\eta$. Given $\eta\in D$, let $c^i_\eta = x^i_\eta + \sum_{j\in J(\eta)}C_{(\eta,j)}\varphi^i_\eta$ be the consumption bundle chosen by agent i at η. Using equation (3), we have that

$$\gamma^i_\eta p_\eta(\sigma W_\eta - c^i_\eta) \ge v^i_\eta(\sigma W_\eta - c^i_\eta) \ge u^i_\eta(\sigma W_\eta) - u^i_\eta(c^i_\eta) \ge u^i_\eta(\sigma W_\eta) - u^i_\eta(W_\eta) > 0.$$

Therefore, $\gamma^i_\eta\|p_\eta\|_\Sigma \ge \underline{\pi}_\eta$ and, at every node $\eta\in\mathrm{Eff}(\Theta)$, since $\Omega(\eta)\neq\emptyset$ and $\min_{j\in\Omega(\eta)}\ \|C_{(\eta,j)}\|_\Sigma < \Psi_\eta$, there exists $j\in\Omega(\eta)$ such that

$$\gamma^i_\eta(p_\eta C_{(\eta,j)} - q_{(\eta,j)}) \le \gamma^i_\eta p_\eta C_{(\eta,j)} - \sum_{\mu\in\eta^+}\gamma^i_\mu F_{(\mu,j)}(p_\mu) < 0, \tag{6}$$

where the last inequality follows from the definition of the upper bound of collateral requirements. Finally, using the Lemma 1 in the Appendix, we conclude that agent i's problem does not have a

[6]Given a concave function $f: X\subset\mathbb{R}^n\to\mathbb{R}\cup\{-\infty\}$ and $x\in X$, the super-differential of f, $\partial f(x)$, is defined as the set of points $p\in X$, called super-gradients, such that $f(y)-f(x)\le p(y-x)$, $\forall y\in X$.

solution, contradicting the optimality of $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$ under prices $(p, q) \in \Pi$. $\square$

Note that, by construction, upper bounds on collateral requirements, $(\Psi_\eta)_{\eta \in \mathrm{Eff}(\Theta)}$, depend only on the primitives of the economy and, for computational objectives, can be easily found.

Also, given any plan of prices, when collateral requirements are not high enough in the sense of the Theorem above, either there is no solution for individual's problem or the associated optimal plans are not physically feasible. In fact, there are cases in which a solution for the individual's problem exists independently of the size of collateral bundles. More precisely, given collateral bundles $(C_{(\xi,j)})_{(\xi,j)\in J(D)}$, it is always possible to find strictly positive prices $(p, q) \in \Pi$ such that, for any $(\xi, j) \in J(D)$, $p_\xi C_{(\xi,j)} - q_{(\xi,j)} > 0$. Therefore, at these prices (p, q), the set of budget feasible allocations is compact in the product topology and, when U^i is *continuous*, there is a solution for agent i's problem. However, when collateral guarantees satisfy the conditions of the Theorem above, our main result ensure that this solution is not physically feasible.

4. Extensions

On endogenous effectiveness.

For each unit of asset sold in our model, we assume that the amount of payments besides the collateral guarantees is independent of the borrowers and does not depend on the history of default. This assumption allow us to pool the debt contracts into derivatives following a trivial pass-through structure identifying both prices and payments of primary (debt) markets with those of secondary (investment) markets.

However, our analysis may be extended for enforcement mechanisms in which the effectiveness is an endogenous and personalized variable. Consider a dynamic general equilibrium model with infinite horizon in which, at any node and for every agent, the access to credit markets and/or the available endowments depend on previous payments. Thus, in this new framework there may be endogenous incentives inducing borrowers to deliver payments larger than the depreciated value of collateral requirements. Also suppose that financial markets still preserve some features from our original model. That is, each type of credit contract is securitized into only one passthrough derivative, primitive and derivative prices are identified, and lenders perfectly foresee the payments of derivatives. Specifically for this last feature, suppose that, in case of default, agents advance any payment in addition to the depreciated collateral as a *percentage* of the remaining debt, facing payment functions with analogous specification of $(F_{(\mu,j)})_{(\mu,j)\in J^+(D)}$.

In this new context, under hypotheses on individual' characteristics analogous to Assumptions A1-A2, suppose that the following properties hold:

(i) For any plan of prices, individual optimal allocations satisfy inequalities (3)-(5);
(ii) If unitary loan prices persistently exceed the associated collateral value, there is no finite optimum for individual problems.

Then, lenders expect persistent equilibrium payments in addition to collateral guarantees only when these collateral requirements are high enough (in a sense analogous of our Theorem).

Therefore, a natural question arise. When does an economy satisfies properties (i) and (ii)? It follows from Lemma 2 that any convex model satisfies property (i).[7] However, some enforcement mechanisms may induce non-convex budget sets. Even in these cases, our arguments can be remade if these non-convexities involve only the borrowers' deliveries.[8]

On the other hand, the proof of property (ii) (see Lemma 1) relies only on the availability of new credits for agents. Thus, (ii) holds unless there is some restriction on the short sales. Indeed, in such a case, agents can not take advantage of the high value of loans (relative to collateral values) by successively increasing their debts. This case is specially important for models in which the access to credit markets depends on the history of default. However, even for such a case and for particular types of preferences, property (ii) can be proved using an alternative argument on Lemma 1. In fact, suppose that, given $\xi \in D$, in addition to the requirement of constituting collateral guarantees, if an agent payed all his debts at nodes $\mu \leq \xi$, then he is not debt constrained at ξ. Also, assume that the preferences of an agent h satisfy Assumption A2 and that

$$\lim_{b\in\mathbb{R}^L_{++}:\|b\|_\Sigma\to+\infty} u^h_\xi(b) = +\infty.$$

Then, to ensure property (ii), we can reprove Lemma 1 showing that, given any candidate to optimal plan, it is always possible to improve agent h's utility, a contradiction. Essentially, this is possible because both $u^h_\xi(b) \to +\infty$ as $\|b\|_\Sigma$ increases, and loans values are greater than those of the associated collateral bundles.

Note that, in any model in which effectiveness is an endogenous variable, the mere presence of additional enforcement mechanisms does not necessarily eliminate equilibrium existence. However, if our results are valid in such a context, lower these requirements are, lower is the maximum persistently effectiveness that additional enforcement mechanisms can induce in equilibrium.

About non-persistent effectiveness.

Our main result relies on the incompatibility between low collateral requirements and *persistent* payments in addition to the value of collateral guarantees. However, when effectiveness is endogenous, our result may still hold even when additional enforcement mechanisms are effective *only* on an asset $j \in J(\xi)$ at a node $\mu \in \xi^+$. In fact, if non-arbitrage conditions imply that, to have a finite optimum at prices (p, q), the value of collateral requirements need to be greater than those associated with the loans, our results can be improved. Essentially, if we suppose that lenders expect payments besides the values of collateral guarantees at $\mu \in \xi^+$, and collateral bundles associated to asset j are not high enough at ξ, we obtain a contradiction only by using property (i) above. Indeed, if there is a finite optimum at prices (p, q), then $p_\xi C_{(\xi,j)} - q_{(\xi,j)} > 0$, which is incompatible with low collateral requirements when property (i) holds, as proved in the theorem.

[7]A model is *convex* when agents' objective functions are concave and, for each vector of prices, budget sets are convex.

[8]Technically, in this case, the arguments in the proof of Lemma 2 can be remade by redefining the truncated problem $(P^{i,T})$ in such form that, for any $\eta \in D^T$, variables φ_η are fixed and equal to the optimal choices φ^i_η.

As a matter of example, suppose that, in case of default on asset $j \in J(\xi)$ at a node $\mu \in \xi^+$, not only the depreciated collateral bundle can be seized but also fractions of endowments, $\alpha^w_{\mu,j} \in [0,1]$, of autonomous consumption bundles, $\alpha^x_{\mu,j} \in [0,1]$, and of long financial positions, $\alpha^\theta_{\mu,j} \in [0,1]$, where $\sum_{j\in J(\mu^-)} \alpha^s_{\mu,j} \leq 1$, for any $s \in \{w, x, \theta\}$. More precisely, if an agent h short-sales $\varphi^h_{(\xi,j)}$ units of an asset j at ξ, then he pays at μ an amount

$$\min\left\{ p_\mu A_{(\mu,j)}\varphi^h_{(\xi,j)},\, p_\mu Y_\mu C_{(\xi,j)}\varphi^h_{(\xi,j)} + \alpha^w_{\mu,j} p_\mu w^h_\mu + \alpha^x_{\mu,j} p_\mu Y_\mu x^h_\xi + \alpha^\theta_{\mu,j} \sum_{\{j'\in J(\xi); j'\neq j\}} N_{\mu,j'}(p)\theta^h_{(\xi,j')} \right\},$$

where x^h_ξ is the autonomous consumption at ξ, $(\theta^h_{(\xi,j)}, j \in J(\xi))$ are the quantities of derivatives bought by agent h at ξ, and for each $j' \in J(\xi)$,

$$N_{\mu,j'}(p) = p_\mu Y_\mu C_{(\mu,j')} + \widehat{\lambda}_{(\mu,j')}\left[p_\mu A_{(\mu,j')} - p_\mu Y_\mu C_{(\mu,j')}\right]^+$$

is the expected payment of asset j' at node μ. Note that, payments in addition to the depreciated collateral value are advanced as an endogenous percentage $\widehat{\lambda}_{(\mu,j')} \in [0,1]$ of the remaining debt.

In this situation, independently of prices or physical-financial allocations, borrowers always have solvency to pay their commitments. Indeed, in case of default, markets will seize fractions of resources that borrowers already have.

About finite lived agents.

Our main result must not hold for *finite horizon* economies. In fact, the scarcity of collateral requirements must ensure the existence of equilibrium in these economies when agents are rational, even with additional effective enforcements. Thus, without losing equilibrium existence, other effective default enforcements may be added to the seminal model of passtrough securities of Geanakoplos and Zame (2002) or to the Steinert and Torres-Martínez's (2007) model of Collateralized Loan Obligations.

On bounded rationality.

In collateralized financial markets, we prove that when infinite-lived agents are rational—in the sense that they perfect foresee future prices and effectiveness of default enforcement mechanisms—any persistently effective additional mechanisms jointly with not large enough collateral guarantees imply in the nonexistence of a physically feasible solution for individual's problem.

Allowing weaker requirements of rationality, Daher, Martins-da-Rocha, Páscoa and Vailakis (2006) show that, even in the presence of utility penalties, the collateralization of debts solves the problems associated to the existence of temporary equilibrium in a two-period economy with default. The main idea is that, independently of the support of individual beliefs about future prices and future states of nature, borrowers hold collateral requirements and to foreclosure their debts it is always possible to deliver *only* the depreciated value of these bundles. Thus, errors in forecasts of future prices do not induce solvency problems in the economy. However, this is a particularity of the additional enforcement mechanism given by utility penalties, since any agent may choose to internalize the associated penalties when his resources are insufficient to honor his financial obligations. In fact, when agents are finite-lived and take into account expectations about future effectiveness analogous to those addressed here, individual's problem may not have a solution. Intuitively, errors in future forecasts of the effectiveness still may lead to solvency problems.

Long-lived assets and endogenous collateral.

Our analysis also holds when long-lived real assets are available for trading. Essentially, non-arbitrage conditions associated to individual's problem are still valid (see Araujo, Páscoa and Torres-Martínez (2007)). Finally, if we want collateral requirements to become endogenous, as in Geanakoplos and Zame (2002), a pool of financial contracts can be offered at each node, with the same real promises but with different associated collateral bundles. Thus, the choice of financial contracts traded by borrowers induce an endogenous choice of the associated collateral. However, it is important to be careful with the size of the available collateral requirements, since individual's optimality may become incompatible with commodity market feasibility.

Appendix

In a context of collateralized assets and linear utility penalties for default, Páscoa and Seghir (2006) show that Ponzi schemes could be implemented if there exists a subtree $D(\xi)$ such that, for every node $\mu \geq \xi$, there is always some asset $j \in J(\mu)$ whose price exceeds the respective collateral value, $p_\mu C_{(\mu,j)} - q_{(\mu,j)} < 0$ (see Remark 3.1 in Páscoa and Seghir (2006)). In such event, the individual's problem does not have a finite solution. In our context, weaker conditions ensure the inexistence of a solution for the individual's problem.

LEMMA 1. *Assume that, given $x \in \mathbb{R}_+^{L\times D}$, if $U^i(x)$ is finite, then $U^i(y) > U^i(x)$ for any $y > x$. Also, suppose that additional enforcement mechanisms are persistently effective in a path $\Theta = (\mu_n; n \in \mathbb{N})$ such that, for any $\eta \in \mathrm{Eff}(\Theta)$, there exists $j \in J(\eta)$ for which $p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0$. Then, agent i's individual problem does not have a finite solution, otherwise, Ponzi schemes could be implemented.*

PROOF. Assume there is a budget feasible plan for agent i, $(x^i, \theta^i, \varphi^i)$, that gives a finite optimum. Under the monotonicity condition stated in the Lemma, $p_\eta \gg 0$ for every node $\eta \in D$. For each $\eta \in \mathrm{Eff}(\Theta)$, let $J^1(\eta) = \{j \in J(\eta) : p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0\}$. Now, consider the allocation $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi\in D}$, with

$$\left((x_\mu, \theta_\mu, \varphi_\mu);\, (\theta_\eta, \varphi_{(\eta,j)})\right)_{\mu\notin \mathrm{Eff}(\Theta),\, \eta\in \mathrm{Eff}(\Theta)} = \left((x^i_\mu, \theta^i_\mu, \varphi^i_\mu);\, (\theta^i_\eta, \varphi^i_{(\eta,j)})\right)_{\mu\notin \mathrm{Eff}(\Theta),\, \eta\in \mathrm{Eff}(\Theta)},\quad \forall j \in J(\eta)\setminus J^1(\eta)$$

and

$$\begin{aligned}
\varphi_{(\eta,j)} &= \varphi^i_{(\eta,j)} + \delta_\eta, \quad \forall \eta \in \mathrm{Eff}(\Theta),\, \forall j \in J^1(\eta),\\
x_{(\eta,l)} &= x^i_{(\eta,l)} + \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j\in J^1(\eta)} \left(q_{(\eta,j)} - p_\eta C_{(\eta,j)}\right)\delta_\eta, \quad \forall l \in L, \text{ if the node } \eta = \mu_1,\\
x_{(\eta,l)} &= x^i_{(\eta,l)} + \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j\in J^1(\eta)} \left(q_{(\eta,j)} - p_\eta C_{(\eta,j)}\right)\delta_\eta\\
&\qquad - \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j\in J^1(\eta^-)} p_\eta A_{(\eta,j)}\,\delta_{\eta^-},\ \forall \eta \in \mathrm{Eff}(\Theta)\setminus\{\mu_1\}, \forall l \in L,
\end{aligned}$$

where the plan $(\delta_\eta)_{\eta\in \mathrm{Eff}(\Theta)}$ is chosen in such form that the following conditions hold,

$$\sum_{j\in J^1(\mu_1)} \left(q_{(\mu_1,j)} - p_{\mu_1} C_{(\mu_1,j)}\right)\delta_{\mu_1} > 0, \tag{7}$$

$$\sum_{j\in J^1(\eta)} \left(q_{(\eta,j)} - p_\eta C_{(\eta,j)}\right)\delta_\eta > \sum_{j\in J^1(\eta^-)} p_\eta A_{(\eta,j)}\,\delta_{\eta^-}, \quad \forall \eta \in \mathrm{Eff}(\Theta)\setminus\{\mu_1\}. \tag{8}$$

By the definition of $\mathrm{Eff}(\Theta)$, it follows that $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$ is budget feasible at prices (p, q). Moreover, equations above show that Ponzi schemes are possible at prices (p, q). In fact, agent i increases his borrowing at μ_1 and pays his future commitments either by using new credit at the nodes in which there is effectiveness or by delivering depreciated collateral guarantees for the nodes $\mu \notin \mathrm{Eff}(\Theta)$ such that $\mu^- \in \mathrm{Eff}(\Theta)$. It follows that $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$ improves the utility level of agent i, contradicting the optimality of $(x^i, \theta^i, \varphi^i)$. $\square$

The following result and its demonstration are analogous to Proposition 1 in Araujo, Páscoa and Torres-Martínez (2007). However, as slight modifications are necessary we present the whole proof for the readers.

Lemma 2. *Let $(p, q) \in \Pi$ and fix a budget and physically feasible plan $z^i := (x^i, \theta^i, \varphi^i) \in \mathbb{E}$. Under Assumptions A1 and A2, if z^i is an optimal allocation for agent i's problem at prices (p, q), then for every $\eta \in D$, the function u^i_η is super-differentiable at the point $c^i_\eta := x^i_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi^i_{(\eta,j)}$, there are multipliers $\gamma^i_\eta \in \mathbb{R}_{++}$ and super-gradients $v^i_\eta \in \partial u^i_\eta \left(c^i_\eta\right)$ such that, for each $j \in J(\eta)$,*

$$\gamma^i_\eta p_\eta \geq v^i_\eta + \sum_{\mu \in \eta^+} \gamma^i_\mu p_\mu Y_\mu, \tag{9}$$

$$\gamma^i_\eta q_{(\eta,j)} \geq \sum_{\mu \in \eta^+} \gamma^i_\mu F_{(\mu,j)}(p_\mu). \tag{10}$$

Also, the plan of multipliers $(\gamma^i_\eta)_{\eta \in D}$ satisfy

$$\gamma^i_\eta p_\eta W^i_\eta \leq \sum_{\eta \in D} u^i_\eta(c^i_\eta). \tag{11}$$

Proof. Given $T \in \mathbb{N}$, define $D_T = \{\eta \in D : t(\eta) = T\}$ and $D^T = \{\eta \in D : \eta \in \bigcup_{k=0}^{T} D_k\}$. For any $\eta \in D$, let $Z(\eta) = \mathbb{R}^L_+ \times \mathbb{R}^{J(\eta)}_+ \times \mathbb{R}^{J(\eta)}_+$. For convenience of notations, let $z_{\xi_0^-} := 0 \in Z(\xi_0^-)$, where $Z(\xi_0^-) := \mathbb{R}^L_+$. Consider the optimization problem:

$$(P^{i,T}) \qquad \begin{array}{ll} \max & \sum\limits_{\eta \in D^T} u^i_\eta \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)}\right) \\ \text{s.t.} & \left\{ \begin{array}{lll} z_\eta := (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta) & & \forall\, \eta \in D^T, \\ g^i_\eta(z_\eta, z_{\eta^-}; p, q) & \leq & 0, \quad \forall\, \eta \in D^T, \\ x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} & \leq & 2W_\eta, \quad \forall\, \eta \in D^T, \\ z_\eta & \in & [0, z^i_\eta], \ \forall\, \eta \in D_T. \end{array} \right. \end{array}$$

where the inequality $g^i_\eta(z_\eta, z_{\eta^-}; p, q) \leq 0$ represents the budget constraint at node η, that is, inequality (1) or (2), and given $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, the interval $[x, y] := \{z \in \mathbb{R}^m : \exists a \in [0, 1],\ z = ax + (1-a)y\}$. It follows from the existence of an optimal individual plan at prices (p, q) that there exists a solution for $(P^{i,T})$, namely $(z^{i,T}_\eta)_{\eta \in D^T}$.[9]

[9]In fact, define a new problem $(\tilde{P}^{i,T})$,

$$(\tilde{P}^{i,T}) \qquad \begin{array}{ll} \max & \sum\limits_{\eta \in D^T} u^i_\eta \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)}\right) \\ \text{s.t.} & \left\{ \begin{array}{lll} z_\eta := (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta) & & \forall\, \eta \in D^T, \\ g^i_\eta(z_\eta, z_{\eta^-}; p, q) & \leq & 0, \quad \forall\, \eta \in D^T, \\ x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} & \leq & 2W_\eta, \quad \forall\, \eta \in D^T, \\ z_\eta & \in & [0, z^i_\eta], \ \forall\, \eta \in D_T, \\ \text{If} \quad q_{(\eta,j)} = 0 & \text{then} & \theta_{(\eta,j)} = 0. \end{array} \right. \end{array}$$

Under Assumption A2 the objective function on $(\tilde{P}^{i,T})$ is continuous, and the set of admissible allocations is compact in $\prod_{\eta \in D^T} Z(\eta)$. Note that, to ensure this it is necessary to have non-zero collateral requirements, otherwise, long and short positions are unbounded.

Given $\eta \in D$, define the concave function $\nu^i_\eta : \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)} \to \mathbb{R} \cup \{-\infty\}$ as

$$\nu^i_\eta(z_\eta) = \begin{cases} u^i_\eta\left(x_\eta + \sum_{j\in J(\eta)} C_{(\eta,j)}\varphi_{(\eta,j)}\right) & \text{if } x_\eta + \sum_{j\in J(\eta)} C_{(\eta,j)}\varphi_{(\eta,j)} \geq 0; \\ -\infty & \text{otherwise.} \end{cases},$$

where $z_\eta = (x_\eta, \theta_\eta, \varphi_\eta)$. It follows that, for any $T \geq 1$, $\sum_{\eta\in D^T} \nu^i_\eta(z^{i,T}_\eta) \leq \sum_{\eta\in D} \nu^i_\eta(z^i_\eta)$.[10]

For each $\eta \in D$ and $\gamma_\eta \in \mathbb{R}_+$, define $\mathcal{L}^i_\eta(\cdot, \gamma; p, q) : \mathbb{Z}(\eta) \times \mathbb{Z}(\eta^-) \to \mathbb{R}$ as

$$\mathcal{L}^i_\eta(z_\eta, z_{\eta^-}, \gamma_\eta;\, p, q) = \nu^i_\eta(z_\eta) - \gamma_\eta\, g^i_\eta(z_\eta, z_{\eta^-}; p, q).$$

Given $T \in \mathbb{N}$, for each $\eta \in D^{T-1}$, define the set $\Xi^T(\eta)$ as the family of allocations $(x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta)$ that satisfies $x_\eta + \sum_{j\in J(\eta)} C_{(\eta,j)}\varphi_{(\eta,j)} \leq 2W_\eta$. Also, for any $\eta \in D_T$, let $\Xi^T(\eta)$ be the set of allocations $(x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta)$ that satisfies both $x_\eta + \sum_{j\in J(\eta)} C_{(\eta,j)}\varphi_{(\eta,j)} \leq 2W_\eta$ and $(x_\eta, \theta_\eta, \varphi_\eta) \in [0, z^i_\eta]$. Let $\Xi^T := \prod_{\eta\in D^T} \Xi^T(\eta)$.

It follows from Rockafellar (1997, Theorem 28.3), that there exist non-negative multipliers $(\gamma^{i,T}_\eta)_{\eta\in D^T}$ such that the following saddle point property holds,

$$\sum_{\eta\in D^T} \mathcal{L}^i_\eta(z_\eta, z_{\eta_-}, \gamma^{i,T}_\eta;\, p, q) \leq \sum_{\eta\in D^T} \mathcal{L}^i_\eta(z^{i,T}_\eta, z^{i,T}_{\eta_-}, \gamma^{i,T}_\eta;\, p, q), \quad \forall (z_\eta)_{\eta\in D^T} \in \Xi^T, \tag{12}$$

and $\gamma^{i,T}_\eta g^i_\eta(z^{i,T}_\eta, z^{i,T}_{\eta_-}; p, q) = 0$.

Claim A. *For each $\mu \in D$, the sequence $(\gamma^{i,T}_\mu)_{T\geq t(\mu)}$ is bounded. Moreover, given $T > t(\mu)$,*

$$\nu^i_\mu(a_\mu) - \nu^i_\mu(z^i_\mu) \leq \left(\gamma^{i,T}_\mu \nabla_1 g^i_\mu(p,q) + \sum_{\eta\in\mu^+} \gamma^{i,T}_\eta \nabla_2 g^i_\eta(p,q)\right) \cdot (a_\eta - z^i_\eta) + \sum_{\xi\in D\setminus D^{T-1}} \nu^i_\xi(z^i_\xi), \quad \forall\, a_\mu \in \Xi^T(\mu),$$

where, for any $\eta \in D$, the vector $(\nabla_1 g^i_\eta(p,q), \nabla_2 g^i_\eta(p,q))$ is defined by

$$\begin{aligned} \nabla_1 g^i_\eta(p,q) &= \left(p_\eta, q_\eta, (p_\eta C_{(\eta,j)} - q_{(\eta,j)})_{j\in J(\eta)}\right), \\ \nabla_2 g^i_\eta(p,q) &= \left(-p_\eta Y_\eta, (F_{(\eta,j)})_{j\in J(\eta)}, (p_\eta Y_\eta C_{(\eta,j)} - F_{(\eta,j)})_{j\in J(\eta)}\right). \end{aligned}$$

Proof. Given $t \leq T$, substitute the following allocation in inequality (12)

$$z_\eta = \begin{cases} (W^i_\eta, 0, 0), & \forall\, \eta \in D^{t-1}, \\ (0,0,0), & \forall\, \eta \in D^T \setminus D^{t-1}. \end{cases}$$

We have:

$$\sum_{\eta\in D_t} \gamma^{i,T}_\eta p_\eta W^i_\eta \leq \sum_{\eta\in D^T} \nu^i_\eta(z^{i,T}_\eta) \leq \sum_{\eta\in D} \nu^i_\eta(z^i_\eta). \tag{13}$$

Assumptions A1 ensure that, for each $\eta \in D$, $\min_{l\in L} W^i_{(\eta,l)} > 0$. Also, Assumption A2 implies that $\|p_\eta\|_\Sigma > 0$, guaranteing that, for each $\mu \in D$, the sequence $(\gamma^{i,T}_\mu)_{T>t(\mu)}$ is bounded.

On the other hand, given $(z_\eta)_{\eta\in D^T} \in \Xi^T$, using (12), we have that

$$\sum_{\eta\in D^T} \mathcal{L}^i_\eta(z_\eta, z_{\eta^-}, \gamma^{i,T}_\eta; p, q) \leq \sum_{\eta\in D} \nu^i_\eta(z^i_\eta).$$

Thus, there is a solution $(z^{i,T}_\eta)_{\eta\in D^T}$. Moreover, this solution for $(\tilde{P}^{i,T})$ is also an optimal choice for $(P^{i,T})$. Essentially, the existence of a *finite* optimum at prices (p,q) for the agent i's problem ensure that, when $q_{(\eta,j)} = 0$, the payments $F_{(\mu,j)}(p_\mu)$ must be equal zero, for each $\mu \in \eta^+$. Thus, when $q_{(\eta,j)} = 0$, choosing positives amounts of $\theta_{(\eta,j)}$ does not induce any gains.

[10] Note that, otherwise, agent i improve his utility in D choosing the allocation $(z^{i,T}_\eta)_{\eta\in D^T}$ in the sub-tree D^T, without making any (physical or financial) trade after the nodes with date T.

Thus, fix $\mu \in D^{T-1}$ and $a_\mu \in \Xi^T(\mu)$. If we evaluate inequality above in

$$z_\eta = \begin{cases} z^i_\eta, & \forall\, \eta \neq \mu, \\ a_\mu, & \text{for } \eta = \mu, \end{cases}$$

we obtain

$$\nu^i_\mu(a_\mu) - \gamma^{i,T}_\mu g^i_\mu(a_\mu, z^i_{\mu^-}; p, q) - \sum_{\eta \in \mu^+} \gamma^{i,T}_\eta g^i_\eta(z^i_\eta, a_\mu; p, q) \leq \nu^i_\mu(z^i_\mu) + \sum_{\eta \in D \setminus D^T} \nu^i_\eta(z^i_\eta). \tag{14}$$

Since functions $(g^i_\xi(\cdot\,; p, q); \xi \in D)$ are affine, we have

$$\begin{aligned} g^i_\mu(a_\mu, z^i_{\mu^-}; p, q) &= \nabla_1 g^i_\mu(p, q) \cdot a_\mu - p_\mu \omega^i_\mu + \nabla_2 g^i_\mu(p, q) \cdot z^i_{\mu^-} \\ g^i_\eta(z^i_\eta, a_\mu; p, q) &= \nabla_1 g^i_\eta(p, q) \cdot z^i_\eta - p_\eta \omega^i_\eta + \nabla_2 g^i_\eta(p, q) \cdot a_\mu, \quad \forall \eta \in \mu^+. \end{aligned}$$

Also, budget feasibility of $(z^i_\eta)_{\eta \in D}$ at prices (p, q), jointly with monotonicity of preferences, ensure that,

$$\begin{aligned} -p_\mu \omega^i_\mu + \nabla_2 g^i_\mu(p, q) \cdot z^i_{\mu^-} &= -\nabla_1 g^i_\mu(p, q) \cdot z^i_\mu, \\ \nabla_1 g^i_\eta(p, q) \cdot z^i_\eta - p_\eta \omega^i_\eta &= -\nabla_2 g^i_\eta(p, q) \cdot z^i_\mu, \quad \forall\, \eta \in \mu^+. \end{aligned}$$

Therefore,

$$\gamma^{i,T}_\mu g^i_\mu(a_\mu, z^i_{\mu^-}; p, q) + \sum_{\eta \in \mu^+} \gamma^{i,T}_\eta g^i_\eta(z^i_\eta, a_\mu; p, q) = \left(\gamma^{i,T}_\mu \nabla_1 g^i_\mu(p, q) + \sum_{\eta \in \mu^+} \gamma^{i,T}_\eta \nabla_2 g^i_\eta(p, q) \right) \cdot (a_\mu - z^i_\mu).$$

Using (14), we conclude the proof. ⊠

Since D is countable and, for any node η, the sequence $(\gamma^{i,T}_\eta)_{T \geq t(\eta)}$ is bounded, using Tychonoff Theorem (see Aliprantis and Border (1999, Theorem 2.57)), there is a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers, $(\gamma^i_\eta)_{\eta \in D}$, such that, for each $\eta \in D$, $\lim_{k \to \infty} \gamma^{i,T_k}_\eta = \gamma^i_\eta$ and

$$\gamma^i_\eta\, g^i_\eta(z^i_\eta, z^i_{\eta_-}; p, q) = 0,$$

where, as we said above, the last equation follows from the strictly monotonicity of u^i_η. Moreover, taking the limit as T goes to infinity in inequality (13) we obtain that

$$\sum_{\eta \in D_t} \gamma^i_\eta p_\eta W^i_\eta \quad \leq \quad \sum_{\eta \in D} \nu^i_\eta(z^i_\eta), \quad \forall t \geq 0. \tag{15}$$

Therefore, equation (11) follows.

Since for any $\eta \in D$, $\Xi^{s_1}(\eta) = \Xi^{s_2}(\eta)$ when $\min\{s_1, s_2\} > t(\eta)$, it follows from the inequality in the statement of Claim above, taking the limit as T goes to infinity, that

$$\nu^i_\eta(a_\eta) - \nu^i_\eta(z^i_\eta) \quad \leq \quad \left(\gamma^i_\eta \nabla_1 g^i_\eta(p, q) + \sum_{\mu \in \eta^+} \gamma^i_\mu \nabla_2 g^i_\mu(p, q) \right) \cdot (a_\eta - z^i_\eta), \quad \forall\, a_\eta \in \Xi^{t(\eta)+1}(\eta). \tag{16}$$

Thus,

$$\left(\gamma^i_\eta \nabla_1 g^i_\eta(p, q) + \sum_{\mu \in \eta^+} \gamma^i_\mu \nabla_2 g^i_\mu(p, q) \right) \in \partial \left(\nu^i_\eta + \delta_{Z_1(\eta)} + \delta_{Z_2(\eta)} \right)(z^i_\eta),$$

where the functions $\delta_{Z_h(\eta)} : \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)} \to \mathbb{R} \cup \{-\infty\}$, $h \in \{1, 2\}$, satisfy

$$\delta_{Z_1(\eta)}(x_\eta, \theta_\eta, \varphi_\eta) \quad = \quad \begin{cases} 0, & \text{if } (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta), \\ -\infty, & \text{otherwise.} \end{cases}$$

$$\delta_{Z_2(\eta)}(x_\eta, \theta_\eta, \varphi_\eta) \quad = \quad \begin{cases} 0, & \text{if } x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta, \\ -\infty, & \text{otherwise.} \end{cases}$$

where $z_\eta = (x_\eta, \theta_\eta, \varphi_\eta) \in \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)}$. Since the plan $(z^i_\eta)_{\eta\in D}$ is physically feasible, there exists a neighborhood V of z^i_η such that $\delta_{Z_2(\eta)}(b) = 0$ for every $b \in V$. Then, we have that $\partial\delta_{Z_2(\eta)}(z^i_\eta) = \{0\}$. Also, it follows by Theorem 23.8 and 23.9 in Rockafellar (1997), that there exists $v^i_\eta \in \partial u^i_\eta(c^i_\eta)$ and $\kappa^i_\eta \in \partial\delta_{Z(\eta)}(x^i_\eta, \theta^i_\eta, \varphi^i_\eta)$ such that

$$\gamma^i_\eta \nabla_1 g^i_\eta(p,q) + \sum_{\mu\in\eta^+} \gamma^i_\mu \nabla_2 g^i_\mu(p,q) = (v^i_\eta, 0, (C_{(\eta,j)} v^i_\eta)_{j\in J(\eta)}) + \kappa^i_\eta. \tag{17}$$

Notice that, by definition, for each $z_\eta \geq 0, \kappa \in \partial\delta_{Z(\eta)}(z_\eta) \Leftrightarrow 0 \leq \kappa(y - z_\eta),\ \forall\ y \geq 0$, therefore, $\kappa^i_\eta \geq 0$. Thus, the inequalities stated in the lemma hold from equation (17). On the other hand, strictly monotonicity of function u^i_η, ensure that $v^i_\eta \gg 0$ and, therefore, it follows from (9), that γ^i_η is *strictly positive*. □

REFERENCES

[1] Aliprantis, C.D., K.C. Border (1999): "Infinite dimensional analysis: a hitchhiker's guide," 2nd edn.. Berlin Heidelberg New York: Springer.

[2] Araujo, A., M. Páscoa, and J.P. Torres-Martínez (2007): "Long-lived collateralized assets and bubbles," working paper No 542, PUC-Rio, Brazil.

[3] Araujo, A., M. Páscoa, and J.P. Torres-Martínez (2002): "Collateral avoids Ponzi schemes in incomplete markets," Econometrica, 70, 1613-1638.

[4] Kubler, F. and K. Schmedders (2003): "Stationary equilibria in asset pricing models with incomplete markets," Econometrica, 71, 1767-1793.

[5] Daher, W., V.F. Martins-da-Rocha, M. Páscoa, and Y. Vailakis (2006): "Sequential trading without perfect foresight: the role of default and collateral," working paper FEUNL, Portugal.

[6] Geanakoplos, J. and W.R. Zame (2002):"Collateral and the enforcement of intertemporal contracts," Yale University working paper.

[7] Páscoa, M. and A. Seghir (2006): "Harsh default penalties lead to Ponzi schemes," working paper No. 1, 2006, IFE, American University of Beirut. To appear in Games and Economic Behaviour.

[8] Steinert, M. and J.P. Torres-Martínez (2007): "General equilibrium in CLO markets", Journal of Mathematical Economics, 43, 709-734.

[9] Rockafellar, R.T. (1997): "Convex analysis," *Princeton University Press*, Princeton, New Jersey, USA.

DEPARTMENT OF ECONOMICS, PONTIFICAL CATHOLIC UNIVERSITY OF RIO DE JANEIRO (PUC-RIO)
RUA MARQUÊS DE SÃO VICENTE 225, GÁVEA, 22453-900 RIO DE JANEIRO, BRAZIL.
E-mail address: revilth@econ.puc-rio.br

DEPARTMENT OF ECONOMICS, PONTIFICAL CATHOLIC UNIVERSITY OF RIO DE JANEIRO (PUC-RIO)
RUA MARQUÊS DE SÃO VICENTE 225, GÁVEA, 22453-900 RIO DE JANEIRO, BRAZIL.
E-mail address: jptorres_martinez@econ.puc-rio.br

Departamento de Economia PUC-Rio
Pontifícia Universidade Católica do Rio de Janeiro
Rua Marques de Sâo Vicente 225 - Rio de Janeiro 22453-900, RJ
Tel.(21) 35271078 Fax (21) 35271084
www.econ.puc-rio.br
flavia@econ.puc-rio.br