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Abstract

The Heckscher-Ohlin model with arbitrary number of goods, factors and countries (consumers) and no restrictions on factor trading is shown to be equivalent to an exchange model whose goods are the productive factors while consumer's indirect demands for factors are derived from their actual demands for consumption goods. This equivalence enables one to import properties like the pathconnectedness of the equilibrium manifold, the uniqueness of equilibrium for sufficiently small volumes of trade and discontinuities of equilibrium selection maps for large volumes of trade into the Heckscher-Ohlin model. This equivalence also provides the proper theoretical background to the important but so far purely empirical role played in international trade by the volume of net trades in factor contents.

Keywords: *Heckscher-Ohlin; general equilibrium; international trade; factor content.*

JEL classification numbers: *D51; F11; F14*

1. Introduction

The main goal of this paper is a study of the general version of the Heckscher-Ohlin model with arbitrary (finite) numbers of countries, goods and factors. In the version studied in this paper, factors are freely traded between countries. That model is often interpreted as representing an integrated world economy. The equilibria of the Heckscher-Ohlin model without factor trading that satisfy Deardorff's "lens condition" for Factor Price Equalization ([9] and [13], p. 108) coincide with the equilibria of the unrestricted Heckscher-Ohlin model. Quite a few properties are known for the $2 \times 2 \times 2$ model but few of them hold true for the general version of the model. The only properties that can be considered as firmly established as those satisfied by versions of the general equilibrium model that are sufficiently general to include the unrestricted Heckscher-Ohlin model. This applies to the

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existence and efficiency of equilibria as follows from the work of Arrow, Debreu and McKenzie for example [1, 10, 20]. This is also true of the genericity of regular economies and of the properties of regular economies, the latter properties having been proved for the general equilibrium model with production subject to constant returns to scale by Kehoe, Mas-Colell and Smale [17, 19, 22].

I will show in this paper that the unrestricted Heckscher-Ohlin model satisfies other properties than just existence, efficiency and regularity. Since some of those properties have not yet been proved for general equilibrium models with sufficiently high levels of generality, the strategy adopted in this paper is to prove that the unrestricted Heckscher-Ohlin model is equivalent to an exchange model, the simplest version of a general equilibrium model. More specifically, the goods of that equivalent exchange model are going to be the productive factors of the Heckscher-Ohlin model. This equivalence between the two models will imply that every property of the factor exchange model will have an equivalent formulation for the unrestricted Heckscher-Ohlin model. This equivalence of the two models will actually take two forms. One is quite elementary. The other one is stronger and involves the comparison of the natural projection mappings of those two models.

The factor exchange model considered in this paper that will be shown to be equivalent to the Heckscher-Ohlin model is not to be confused with the neoclassical representation of international trade described for example by Chipman [8] and that consists in an exchange model where countries (consumers) make no differences between consumption goods and factors. These two exchange models are different. They involve different goods and, in theory at least, even the number of their goods are different.

In this paper, I emphasize theoretical properties of the Heckscher-Ohlin that are rarely if at all stated in the literature on international trade. These properties deal with the structure of the equilibrium manifold, which includes its pathconnectedness, the uniqueness and the number of equilibria, the continuity or lack of continuity of equilibrium selections. The uniqueness problem (of equilibrium in the Heckscher-Ohlin model) is also given a complete solution for the two-country case.

This paper is organized as follows. Section 2 is devoted to a brief presentation of the unrestricted Heckscher-Ohlin model. The factor exchange model whose "goods" are the (productive) factors is described in Section 3. Section 4 is devoted to the equivalence between the factor exchange model and the Heckscher-Ohlin model with goods and factors. Section 5 deals with the properties of consumers' demands for factors in the associated factor exchange model. Section 6 is devoted to establishing a number of theoretical properties of the Heckscher-Ohlin model: structure of the equilibrium manifold and its pathconnectedness; uniqueness and multiplicity of equilibrium; the continuity or lack of continuity of equilibrium selections. Section 7 is devoted to the case of transitive preferences represented by utility functions and followed by a brief application to the $2 \times 2 \times 2$ Heckscher-Ohlin model. Concluding comments end the paper with Section 8. Properties that are generally well-known for the two-good and two-factor cases are recalled with the level of generality and rigor appropriate for this paper in Appendix 1. A very useful condition for a map to be a smooth embedding is stated and proved in Appendix 2.

Basic knowledge of smooth manifolds and mappings contained in the first pages of Milnor's marvelous little book [21] is all that is needed for reading this paper. Some valuable geometric insight can also be gained with the help of Guillemin and Pollack's excellent book [15].

2. The Heckscher-Ohlin model

2.1. Goods, factors and prices

There are $k \geq 2$ consumption goods and $\ell \geq 1$ pure primary factors. Goods and factors are freely traded. Prices are all strictly positive and represented by the vector (q, p) for goods and factors respectively. It is often convenient to normalize price vectors, in which case the ℓ-th factor is taken as numeraire, i.e., $p_\ell = 1$. Define $S = \mathbb{R}^{\ell-1}_{++} \times \{1\}$ and $X = \mathbb{R}^k_{++}$ the strictly positive orthant of the goods space $\mathbb{R}^k$. The set $X \times \mathbb{R}^\ell_{++}$ consists of non-normalized price vectors (q, p). The set of numeraire normalized prices is then the Cartesian product $X \times S$. Unless the contrary is explicitly stated, prices are normalized by the numeraire convention.

2.2. Consumption

There are $m \geq 2$ of consumers. The consumption set of every consumer is the strictly positive orthant $X = \mathbb{R}^k_{++}$. Consumer i with $1 \leq i \leq m$ is equipped with a demand function $h_i : X \times \mathbb{R}_{++} \to X$. The following properties are usually considered for demand functions for goods: smoothness (S); Walras law (W); homogeneity of degree zero (H); a boundary assumption (A); the weak axiom of revealed preferences (WARP); the negative definiteness of the truncated Slutsky matrix (ND). (For details, see Appendix, Definition A.1.)

Consumer i is endowed with factors that are represented by the vector $\omega_i \in \mathbb{R}^\ell_{++}$. The m-tuple $\omega = (\omega_i)$ represents the endowments of the m consumers in the economy. The set of possible endowments, also known as the endowment or parameter space, is denoted by $\Omega = (\mathbb{R}^\ell_{++})^m$.

Remark 1. The demand functions considered in this paper include as a special case those that result from the budget constrained maximization of utility functions $u_i : X \to \mathbb{R}$ that satisfy standard assumptions. The approach through demand functions instead of utility functions is more general. In particular, it does not require the transitivity of preference relations.

2.3. Production

There is no joint production of consumption goods. The quantity x^j of good j produced with the inputs $\eta = (\eta^1, \dots, \eta^\ell)$ is a smooth function $x^j = F^j(\eta^1, \dots, \eta^\ell) \geq 0$ that is monotone, homogeneous of degree one and concave.

The factor bundle $b_j(p) \in \mathbb{R}^{\ell}_{++}$ is the unique bundle that minimizes the cost $p^T \eta$ of producing (at least) one unit of good j given the (non-normalized) factor price vector $p \in \mathbb{R}^{\ell}_{++}$. This function can be viewed as the demand function of the productive sector for the factors required for the production of good j. That demand for factors $b_j : \mathbb{R}^{\ell}_{++} \to \mathbb{R}^{\ell}_{++}$ is smooth and, for non-normalized factor prices, homogeneous of degree zero. The associated cost function $\sigma_j(p) = p^T b_j(p)$ is smooth, homogeneous of degree one and concave. (See Appendix, Proposition A.5.) The production matrix for the non-normalized factor price vector $p \in \mathbb{R}^{\ell}_{++}$ is the $\ell \times k$ matrix $B(p) = \begin{bmatrix} b_1(p) & b_2(p) & \dots & b_k(p) \end{bmatrix}$.

2.4. Factor content of goods bundles

The factor content $y \in \mathbb{R}^{\ell}_{++}$ of the goods bundle $x \in \mathbb{R}^{k}_{++}$ is defined as the quantities of all factors that minimize the total cost of producing the consumption goods that make up the bundle x. This factor content depends on the (non-normalized) factor price vector $p \in \mathbb{R}^{\ell}_{++}$ and, using matrix notation, is equal to $y = B(p)\, x$.

2.5. Equilibrium

The 3-tuple $(q, p, \omega) \in X \times S \times \Omega$ (where the price vector (q, p) is numeraire normalized) is an equilibrium of the Heckscher-Ohlin model if there exists a bundle of goods $x \in X$ such that the following equalities are satisfied:

$$\sum_{1 \le i \le m} h_i(q, p^T \omega_i) = x, \tag{1}$$

$$B(p)\, x = \sum_{1 \le i \le m} \omega_i. \tag{2}$$

Equality (1) states that the economy produces enough goods to satisfy total demand. They are represented by the goods bundle $x \in X = \mathbb{R}^{k}_{++}$ that is produced in the economy to satisfy total demand. Equality (2) means that there are enough resources in factors for the production of the goods bundle x.

At the equilibrium (q, p, ω), the final consumption of goods by the m consumers is represented by m-tuple $\big(h_i(q, p^T \omega_i)\big) \in X^m$. This allocation of goods is also known as the equilibrium allocation associated with the equilibrium (q, p, ω).

The equilibrium manifold for the Heckscher-Ohlin model is the subset $\tilde{E}$ of $X \times S \times \Omega$ consisting of equilibria (q, p, ω). The natural projection $\tilde{\pi} : E \to \Omega$ is the restriction to the equilibrium manifold $\tilde{E}$ of the projection map $(q, p, \omega) \to \omega$.

A direct study of the Heckscher-Ohlin model following the approach of [7] through the equilibrium manifold and the natural projection is theoretically possible. It would face, however, a very serious hurdle due to the lack of a natural candidate for the concept of no-trade equilibrium because consumers' factor endowments cannot be consumed. The approach followed in the current paper bypasses this problem by proving the equivalence of the Heckscher-Ohlin model with an exchange model.

3. The associated factor exchange model

Goods for the factor exchange model are the factors of the Heckscher-Ohlin model. There are therefore ℓ "goods" for that model and the ℓ-th good or, more simply, factor is taken as the numeraire.

There are m consumers in the factor exchange model where consumer i's demand function $f_i : S \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{++}$ is defined by

$$f_i(p, w_i) = B(p)\, h_i(B(p)^T p, w_i) \tag{3}$$

where $B(p)$ is the production matrix defined in A.2

An economy for the factor exchange model is defined by a specific value of the endowment vector $\omega = (\omega_i) \in \Omega$.

The pair $(p, \omega) \in S \times \Omega$ is an equilibrium of the factor exchange model if the (equilibrium) equation

$$\sum_{1 \leq i \leq m} f_i(p, p^T \omega_i) = \sum_{1 \leq i \leq m} \omega_i \tag{4}$$

is satisfied.

The "equilibrium manifold" E for the factor exchange model is the subset of $S \times \Omega$ that consists of the equilibria (p, ω). The natural projection for the factor exchange model $\pi : E \to \Omega$ is the restriction to the "equilibrium manifold" E of the projection map $(p, \omega) \to \omega$.

4. Equivalence of the Heckscher-Ohlin and factor exchange models

Equivalence means that, roughly speaking, the two models have the same properties. This equivalence takes two forms. An elementary or weak version uses algebra to show that the equilibrium equation of one model can be reduced to the equilibrium equation of the other model and conversely. A stronger form of equivalence is expressed by way of a commutative diagram of maps that involves the natural projections of the Heckscher-Ohlin model and its associated factor exchange model. That stronger form is necessary if one wants to import properties of the equilibrium manifold and natural projection from the factor exchange model into the Heckscher-Ohlin model.

4.1. Weak equivalence

The two models are weakly equivalent if their equilibrium equations are equivalent. This is expressed by the following Proposition:

Proposition 1. *The triple $(q, p, \omega) \in \mathbb{R}^k_{++} \times S \times \Omega$ is an equilibrium of the Heckscher-Ohlin model if and only if $q = B(p)^T p$ and the pair $(p, \omega) \in S \times \Omega$ is an equilibrium of the associated factor exchange model.*

Necessity. Let $(q, p, \omega) \in X \times S \times \Omega$ be an equilibrium of the Heckscher-Ohlin model. The production of commodity j is a zero-profit operation because of the constant returns to scale. This implies the equality $q_j = b_j(p)^T p$ for $1 \leq j \leq k$. In matrix form, this yields

$$q = B(p)^T p \tag{5}$$

Substituting $B(p)^T p$ to q followed by the matrix multiplication by $B(p)$ of the two sides of equilibrium equation (1) in Section 2.5 yields

$$\sum_{1 \leq i \leq m} B(p)\, h_i(B(p)^T p, p^T \omega_i) = B(p)\, x,$$

which combined with (2) yields

$$\sum_{1 \leq i \leq m} B(p)\, h_i(B(p)^T p, p^T \omega_i) = \sum_{1 \leq i \leq m} \omega_i, \tag{6}$$

which can be rewritten as

$$\sum_{1 \leq i \leq m} f_i(p, p^T \omega_i) = \sum_{1 \leq i \leq m} \omega_i, \tag{7}$$

the (equilibrium) equation for the factor exchange model satisfied by the pair $(p, \omega) \in S \times \Omega$.

Sufficiency. Let $(p, \omega) \in S \times \Omega$ be an equilibrium of the factor exchange model. Define $q = B(p)^T p \in X$ and let

$$x = \sum_{1 \leq i \leq m} h_i(q, p^T \omega_i). \tag{8}$$

Each vector $h_i(q, p^T \omega_i)$ belongs to $X = \mathbb{R}^k_{++}$ as does the sum $x = \sum_{1 \leq i \leq m} h_i(q, p^T \omega_i)$. Left multiplication by $B(p)$ of (8) yields

$$B(p)\, x = B(p) \sum_{1 \leq i \leq m} h_i(q, p^T \omega_i). \tag{9}$$

Since $p \in S$ solves (7) and, therefore, (6), the right-hand side term of (9) is equal to $\sum_{1 \leq i \leq m} \omega_i$, from which follows the equality

$$B(p)\, x = \sum_{1 \leq i \leq m} \omega_i,$$

which is equilibrium equation (2) in Section 2.5. The triple (q, p, ω) is therefore an equilibrium of the Heckscher-Ohlin model. $\square$

Factor content of equilibrium allocations

With the factor content of the goods bundle x defined as equal to $y = B(p)x$ in Section 2.4, the allocation of factor contents for the equilibrium allocation associated with the equilibrium $(q, p, \omega) \in \tilde{E}$ is the m-tuple $\big(B(p) h_i(q, p^T \omega_i)\big) \in (\mathbb{R}^\ell_{++})^m$.

Proposition 2. *Let $(q, p, \omega) \in \tilde{E}$ be an equilibrium of the Heckscher-Ohlin model. The factor content of the corresponding equilibrium allocation is the factor allocation associated with the equilibrium $(p, \omega) \in E$ in the factor exchange model.*

Proof. Follows readily from equality $f_i(p, p^T\omega_i) = B(p)h_i(B(p)^T p, p, p^T\omega_i)$. □

Net trade in factor contents at the equilibrium $(q, p, \omega) \in \tilde{E}$ in the Heckscher-Ohlin model is represented by the vector $\left(f_i(p, p^T\omega_i) - \omega_i\right) \in (\mathbb{R}^\ell)^m$.

4.2. Strong equivalence

Let the maps $\alpha : X \times S \times \Omega \to S \times \Omega$ and $\beta : S \times \Omega \to X \times S \times \Omega$ be defined by $\alpha(q, p, \omega) = (p, \omega)$ and $\beta(p, \omega) = (B(p)^T p, p, \omega)$. It then comes:

Proposition 3.

i) *The maps α and β are smooth;*
ii) $\alpha \circ \beta = \mathrm{id}_{S\times\Omega}$*;*
iii) *The map $\beta : S \times \Omega \to X \times S \times \Omega$ is an embedding;*
iv) *The image $F = \beta(S \times \Omega)$ is a smooth submanifold of $X \times S \times \Omega$ that is diffeomorphic to $S \times \Omega$;*
v) $\alpha(\tilde{E}) = E$ *and* $\beta(E) = \tilde{E}$*;*

Proof. Properties *(i)* and *(ii)* are obvious. It follows from (ii) combined with Lemma B.1 in Appendix B that β is an embedding, which proves (iii). Then, (iv) follows from (ii) combined with the definition of an embedding. Property (v) follows readily from Proposition 1. □

It follows from Proposition 3 that maps $\tilde{\alpha} : \tilde{E} \to E$ and $\tilde{\beta} : E \to \tilde{E}$ can be defined by the same formulas as for the maps α and β. The following Corollary is then obvious.

Corollary 4.
$$\tilde{\alpha} \circ \tilde{\beta} = \mathrm{id}_E \quad ; \quad \tilde{\beta} \circ \tilde{\alpha} = \mathrm{id}_{\tilde{E}} \,.$$

The following Theorem states a property relating the Heckscher-Ohlin model and its associated factor exchange model. This property is also taken as the definition of the strong equivalence of these two models. This concept of strong equivalence is closely related to the equivalence concept for smooth maps of Differential Topology (see for example [14], Chapter III, Definition 1.1. or [5], Definition 5.4.2) because, as we will see shortly, the strong equivalence of the two models implies the equivalence in the sense of Differential Topology of the two natural projections $\pi : E \to \Omega$ and $\tilde{\pi} : \tilde{E} \to \Omega$.

Theorem 5 (and definition of strong equivalence)**.** *The following diagram is commutative:*

$$\begin{tikzcd}
X \times S \times \Omega \arrow[rr, shift right, "\alpha"'] & & S \times \Omega \arrow[ll, shift right, "\beta"'] \\
\tilde{E} \arrow[u, hook] \arrow[rr, shift right, "\tilde{\alpha}"'] \arrow[dr, "\tilde{\pi}"'] & & E \arrow[u, hook] \arrow[ll, shift right, "\tilde{\beta}"'] \arrow[dl, "\pi"] \\
& \Omega &
\end{tikzcd}$$

Proof. Follows readily from the formulas defining the maps α, $\tilde{\alpha}$, β and $\tilde{\beta}$. □

5. Properties of the induced demand functions for factors

The equivalence theorem 5 enables one to establish properties of the Heckscher-Ohlin model from those of the associated factor exchange model. The properties of the latter model depend on the properties of the m-tuple (f_i) of consumers' induced demand functions for factors. The question is therefore how the properties of the induced factor demand function f_i depend on those of consumer i's demand function for consumption goods h_i.

5.1. Smoothness (S) and Walras law (W)

Proposition 6.

i) (S) for $h_i \Longrightarrow$ (S) for f_i;
ii) (W) for $h_i \Longrightarrow$ (W) for f_i.

Proof. (i). Let h_i satisfy (S). The production matrix function $p \to B(p)$ is smooth by Proposition A.7 (i) of the Appendix. The demand function $f_i(p, w_i) = B(p)\, h_i(B(p)^T p, w_i)$ is smooth as being the composition of two smooth functions.

(ii). Let h_i satisfy (W). It then comes

$$p^T f_i(p, w_i) = p^T B(p)\, h_i(B(p)^T p, w_i) = q^T h_i(q, w_i) = w_i. \qquad \square$$

From now on in this paper, all demand functions h_i and, therefore, the induced demand functions f_i, satisfy (S) and (W).

5.2. Boundary behavior (A)

Proposition 7. *(A) for $h_i \Longrightarrow$ (A) for f_i.*

Proof. Factor price vectors are not normalized. Let $(p^t, w_i^t) \in \mathbb{R}^{\ell}_{++}$ be a sequence of non-normalized price and income vectors converging to $(p^0, w_i^0) \in \mathbb{R}^{\ell}_{+} \times \mathbb{R}_{++}$, with some but not all coordinates of the price vector p^0 equal to zero. It follows from Proposition A.5 (v) that, for each good j , with $1 \leq j \leq k$, there is at least one factor κ whose demand $b_j^{\kappa}(p^t)$ tends to $+\infty$.

Let $q^t = B(p^t)^T p^t$. By continuity, the sequence q^t tends to a limit $q^0 \in \mathbb{R}^k_+$, where some coordinates of q^0 may be equal to 0. If none of these coordinates are equal to 0, continuity implies $\lim_{t\to\infty} h_i(q^t, w_i^t) = h_i(q^0, w_i^0) \in \mathbb{R}^k_{++}$. It then follows from

$$f_i(p^t, w_i^t) = B(p^t)\, h_i(B(p^t)^T p^t, w_i^t) = B(p^t)\, h_i(q^t, w_i^t)$$

that $\lim_{t\to\infty} \|f_i(p^t, w_i^t)\|$ is equal to $+\infty$ by Proposition A.5 (v) of the Appendix. If some coordinates of q^0 are equal to 0, it follows from Property (A) that is satisfied by h_i that $\limsup_{t\to\infty} \|h_i(q^t, w_i^t)\| = +\infty$. This implies that the demand for at least one production factor κ must tend to $+\infty$. $\square$

5.3. Weak axiom of revealed preferences (WARP)

Proposition 8. *(WARP) for $h_i \Longrightarrow$ (WARP) for f_i.*

Proof. (i). Let (p, w_i) and (p', w_i') be such that $(p')^T f_i(p, w_i) \leq w_i'$ and $f_i(p, w_i) \neq f_i(p', w_i')$. Assume $p = p'$. Inequality $(p')^T f_i(p, w_i) \leq w_i'$ becomes by (W) $w_i \leq w_i'$. The assumption $f_i(p, w_i) \neq f_i(p, w_i')$ implies $w_i \neq w_i'$, from which follows $w_i < w_i'$. Then, by (W), it comes $p^T f_i(p', w_i') = w_i' > w_i$.

Let now $p \neq p'$. The inequality $(p')^T f_i(p, w_i) \leq w_i'$ can be spelled out as

$$(p')^T B(p)\, h_i(B(p)^T p, w_i) \leq w_i'. \tag{10}$$

For $q = B(p)^T p$ and $q' = B(p')^T p'$, the positivity of matrices $B(p)$ and $B(p')$ and of the demand vector $h_i(q, w_i) \in X$ combined with the (strict) inequality $(p')^T B(p') < (p')^T B(p)$ that is satisfied by Proposition A.5 (i) implies the strict inequality

$$(p')^T B(p')\, h_i(q, w_i) < (p')^T B(p)\, h_i(q, w_i),$$

which, combined with (10), yields

$$(q')^T h_i(q, w_i) < w_i'. \tag{11}$$

This strict inequality implies the inequality $h_i(q, w_i) \neq h_i(q', w_i')$. (Otherwise, equality $h_i(q, w_i) = h_i(q', w_i')$ implies $(q')^T h_i(q, w_i) = (q')^T h_i(q', w_i') = w_i'$ by (W), a contradiction with (11).)

By (WARP) satisfied by h_i, inequality (11) implies the strict inequality

$$q^T h_i(q', w_i') > w_i, \tag{12}$$

which can be rewritten as

$$p^T B(p)\, h_i(q', w_i') > w_i. \tag{13}$$

It follows again from the positivity of matrices $B(p)$, $B(p')$ and the demand vector $h_i(q', w_i') \in X$ that inequality $p^T B(p) \leq p^T B(p')$, a consequence of Proposition A.5 (i), implies inequality

$$p^T B(p) h_i(q', w_i') \leq p^T B(p') h_i(q', w_i').$$

This inequality can be rewritten as

$$q^T h_i(q', w_i') \leq p^T B(p') h_i(q', w_i') = p^T f_i(p', w_i').$$

Combining this inequality with the strict inequality (12) yields

$$p^T f_i(p', w_i') > w_i,$$

which proves (WARP) for f_i. □

5.4. Negative definiteness of the truncated Slutsky matrix (ND)

Proposition 9. *(ND) for $h_i \Longrightarrow$ (ND) for f_i.*

Proof. The factor price vector $p \in \mathbb{R}^{\ell}_{++}$ is not normalized in this part because the computation of Slutsky matrices requires taking derivatives with respect to the prices of all factors including the numeraire. Without price normalization, the factor demand function $f_i(p, w_i)$ is homogeneous of degree zero. The $\ell \times \ell$ matrix $\partial_p f_i(p, w_i)$ consists of the first order derivatives of f_i with respect to the (coordinates of the) price vector p. Similarly, let $\partial_q h_i(q, w_i)$ denote the $k \times k$ matrix of partial derivatives for the goods demand function h_i with respect to the goods price vector $q \in X$. Let $q = B(p)^T p$.
Step 1: Negative definiteness of the restriction of the quadratic form associated with matrix $\partial_q h_i(q, w_i)$ to the hyperplane $\{z \in \mathbb{R}^k \mid z^T h_i(q, w_i) = 0\}$. It follows from Hildenbrand and Jerison [16] that (ND) for h_i is equivalent to the restriction of the quadratic form

$$z \in \mathbb{R}^k \to z^T \partial_q h_i(q, w_i)\, z$$

to the hyperplane $h_i(q, w_i)^\perp = \{z \in \mathbb{R}^k \mid z^T h_i(q, w_i) = 0\}$ being negative definite.
Step 2: $\partial_p f_i(p, w_i) = DB(p)\, h_i(B(p)^T p, w_i) + B(p)\, \partial_p h_i(B(p)^T p, w_i)$. Follows from taking the derivative of the product $f_i(p, w_i) = B(p) h_i(B(p)^T p, w_i)$ with respect to the price vector $p \in \mathbb{R}^{\ell}_{++}$.

Step 3: $\partial_p h_i(B(p)^T p, w_i) = \partial_q h_i(q, w_i)\, B(p)^T$. Application of the chain rule yields

$$\partial_p h_i(B(p)^T p, w_i) = \partial_q h_i(q, w_i)\, D\big(B(p)^T p\big).$$

It then suffices to apply Proposition A.7, (iii).
Step 4: $\partial_p f_i(p, w_i) = DB(p)\, h_i(q, w_i) + B(p)\, \partial_q h_i(q, w_i)\, B(p)^T$. It suffices to substitute the expression obtained in Step 3 in the formula of Step 2.

Step 5: The quadratic form defined $v \in \mathbb{R}^\ell \to v^T DB(p)\, h_i(q, w_i)v$ is negative semi-definite, with rank $\ell - 1$ and $v^T DB(p)\, h_i(q, w_i)v < 0$ for v not collinear with p. The column matrix $B(p)\, h_i(q, w_i)$ is equal to

$$B(p)\, h_i(q, w_i) = \sum_{1 \le j \le \ell} b_j(p)\, h_i^j(q, w_i).$$

Its partial derivative with respect to p (q is considered as fixed and independent of p in this formula) is the $\ell \times \ell$ matrix

$$DB(p)\, h_i(q, w_i) = \sum_{1 \le j \le \ell} (Db_j(p))\, h_i^j(q, w_i).$$

Each square matrix $Db_j(p)$ defines a quadratic form that is negative semidefinite, with rank $k - 1$ and kernel collinear with p by Proposition A.5, (v) and (vii).

The linear combination of these negative semidefinite quadratic forms with the strictly positive coefficients $h_i^j(q, w_i)$, with $1 \le j \le k$, is negative semidefinite and takes a value different from zero. It is therefore strictly negative for any vector $v \in \mathbb{R}^\ell$ that is not collinear with the price vector $p \in \mathbb{R}^\ell_{++}$.
Step 6: $v \in f_i(p, w_i)^\perp$ implies $B(p)^T v \in h_i(B(p)^T p, w_i)^\perp$. The relation $v \in f_i(p, w_i)^\perp$ is equivalent to $v^T f_i(p, w_i) = v^T B(p) h_i(B(p)^T p, w_i) = 0$. This relation is equivalent to $z = B(p)^T v \in h_i(B(p)^T p, w_i)^\perp$.
Step 7: $p^T f_i(p, w_i) \neq 0$ for any $p \in \mathbb{R}^\ell_{++}$. Follows readily from Walras law: $p^T f_i(p, w_i) = w_i \neq 0$.
Step 8: $v^T \partial_p f_i(p, w_i)\, v < z^T \partial_q h_i(q, w_i)\, z$ for any $v \neq 0 \in f_i(p, w_i)^\perp$, $q = B(p)^T p$ and $z = B(p)^T v$. Let $v \neq 0 \in h_i(q, w_i)^\perp$. The vector v is not collinear with p. Assume the contrary. There exists $\lambda \neq 0$ with $v = \lambda p$. Then, it comes $v^T f_i(p, w_i) = \lambda p^T f_i(p, w_i) \neq 0$ by Step 7, a contradiction.

The strict inequality

$$v^T DB(p)\, h_i(q, w_i)\, v < 0$$

then follows from Step 5. The combination with Step 4 implies

$$v^T \partial_p f_i(p, w_i)\, v = v^T DB(p)\, h_i(q, w_i)\, v + v^T B(p)\, \partial_q h_i(q, w_i)\, B(p)^T v,$$

from which follows the strict inequality

$$v^T \partial_p f_i(p, w_i)\, v < z^T \partial_q h_i(q, w_i)\, z. \tag{14}$$

Step 9: The restriction of the quadratic form defined by $\partial_p f_i(p, w_i)$ to the hyperplane $f_i(p, w_i)^\perp$ is negative definite. Follows readily from the strict inequality (14) combined with the negative definiteness of the restriction of the quadratic form defined by $\partial_q h_i(q, w_i)$ to the hyperplane $h_i(q, w_i)^\perp$ proved in Step 1.
Step 10: f_i satisfies (ND). The equivalence of the property stated in Step 9 with (ND) for f_i now follows from Hildenbrand and Jerison [16]. □

6. Applications to the Heckscher-Ohlin model

6.1. Regular and singular economies for the Heckscher-Ohlin model

The properties of the Heckscher-Ohlin model considered in this paper are articulated around the concept of smooth mappings and their critical and regular points and values. More specifically, assume that the natural projection $\tilde{\pi} : \tilde{E} \to \Omega$ is a smooth map between smooth manifolds. (This property will be shown to be true in a moment.)

One defines a regular (resp. critical) point of that map as an element x of $\tilde{E}$ such that the derivative $D_x\tilde{\pi} : T_x(\tilde{E}) \to T_{\tilde{\pi}(x)}(\Omega)$ is (resp. is not) a bijection. The spaces $T_x(\tilde{E})$ and $T_{\tilde{\pi}(x)}(\Omega)$ are the tangent spaces to $\tilde{E}$ and Ω at $x \in \tilde{E}$ and $\tilde{\pi}(x) \in \Omega$ respectively. By definition, a singular value $\omega \in \Omega$ of the map $\tilde{\pi} : \tilde{E} \to \Omega$ is the image of a critical point, i.e., there exists $x \in \tilde{E}$ that is a critical point and such that $\tilde{\pi}(x) = \omega \in \Omega$. Let Σ denote the set of singular values of the projection map $\tilde{\pi} : \tilde{E} \to \Omega$. This set is the image by $\tilde{\pi}$ of the set of critical points.

The element $\omega \in \Omega$ is by definition a regular value of the map $\tilde{\pi} : \tilde{E} \to \Omega$ if it is not a singular value. The set of regular values $\mathcal{R}$ of the map $\tilde{\pi}$ is therefore the complement $\Omega \setminus \Sigma$ of the set of singular values Σ. Note that one often uses the terms of regular and singular economies instead of regular and singular values of the natural projection $\tilde{\pi}$.

6.2. The natural projection as a smooth mapping

The following Proposition describes properties of the Heckscher-Ohlin model with demand functions h_i satisfying no other assumption than (S) and (W).

Proposition 10.

i) *The "equilibrium manifold" of the Heckscher-Ohlin model $\tilde{E}$ is a smooth submanifold of $X \times S \times \Omega$ diffeomorphic to $\mathbb{R}^{\ell m}$.*
ii) *The natural projection for the Heckscher-Ohlin model $\tilde{\pi} : \tilde{E} \to \Omega$ is smooth.*
iii) *The regular and singular values of the natural projection $\tilde{\pi} : \tilde{E} \to \Omega$ are the regular and singular values respectively of the natural projection $\pi : E \to \Omega$ of the associated factor exchange model.*
iv) *The set of factor contents for the equilibrium allocations of the Heckscher-Ohlin model is identical to the set P of equilibrium allocations of the associated factor exchange model.*
v) *The set P is pathconnected.*

Proof. Property (i) follows from Proposition 3 (iv) combined with the property that E, the equilibrium manifold of the associated factor exchange model, is a smooth submanifold of $S \times \Omega$ diffeomorphic to $\mathbb{R}^{\ell m}$ when all factor demand functions f_i satisfy (S) and (W). Properties (ii) and (iii) follow readily from the commutativity of the lower triangle in the diagram of Theorem 5. Property (iv) is essentially a reformulation of Proposition 2. To prove (v), it suffices to observe that the set of no-trade equilibria in the factor exchange model is pathconnected by [7], Proposition 5.2, and that the set P is the image by a continuous map, the natural projection $\pi : E \to \Omega$, of the set of no-trade equilibria. $\square$

Sard's theorem [21] implies that the set of singular values Σ of the natural projection $\tilde{\pi}$ has measure zero in Ω. This set is therefore small from a measure theoretic point of view. But it can still be large in a topological sense as, for example, the set of rational numbers $\mathbb{Q}$ that is dense in $\mathbb{R}$. This does not happen if the set Σ happens to be closed as is the case if the map $\tilde{\pi}$ is proper, i.e., the preimage $\tilde{\pi}^{-1}(K)$ of every compact subset K of Ω is compact. The following Proposition gives us a sufficient condition for the properness of $\tilde{\pi}$:

Proposition 11. *If at least one demand function for goods h_i satisfy (A), the natural projection of the Heckscher-Ohlin model $\tilde{\pi} : \tilde{E} \to \Omega$ is proper.*

Proof. It follows from Theorem 5 that $\tilde{\pi}$ is proper if and only if the natural projection $\pi : E \to \Omega$ is proper. This follows from Proposition 7.1 in [7] if some demand function f_i satisfies the boundary condition (A). It then suffices that some demand function for goods h_i satisfies (A). $\square$

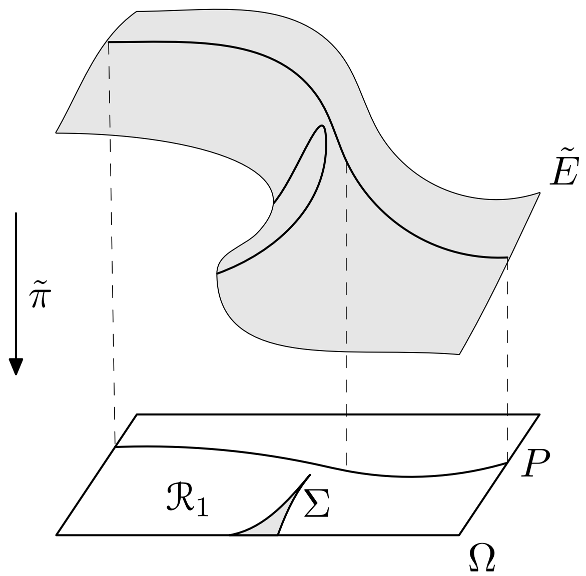

Figure 1: Heckscher-Ohlin model: equilibrium manifold and natural projection

The smooth and proper map $\tilde{\pi} : \tilde{E} \to \Omega$ is therefore a ramified covering of Ω, which is illustrated on Figure 1. As such, it entails quite a few remarkable and economically important properties for the Heckscher-Ohlin model. These properties include in particular the genericity of regular economies (i.e., the set of regular economies $\mathcal{R}$ is open with full measure in Ω) and the finite odd number of equilibria at regular economies. For more details on the derivation of those properties from the smoothness and properness of the natural projection, see [7], Chapter 7.

This theory is often known as the theory of regular economies and was first developed for the exchange model by Debreu and Dierker using a different approach [11, 12].

6.3. Number of equilibria

All demand functions for goods h_i are now assumed to satisfy (S), (W) and (WARP). In addition, at least one demand function h_i satisfies (A) and (ND). Property (WARP) is economically important because it essentially says that the preferences (possibly non transitive) that underlie consumers' demand functions for goods are convex. In other words, (WARP) captures the essence of convexity. Property (ND) satisfied by just one demand function is only a slight strengthening of (WARP) because it amounts to substituting a strict unequal sign to a large one.

Unsurprisingly, much stronger properties of the Heckscher-Ohlin model follow from the assumption of convexity conveyed by (WARP). The number of equilibria associated with the regular economy $\omega \in \mathcal{R}$ becomes related to the location of ω with respect to the set P of factor contents of equilibrium allocations, a set that coincides with the set of equilibrium allocations of the factor exchange model by Proposition 2. More specifically, it comes:

Proposition 12. *The set P of factor contents of equilibrium allocations is contained in one connected component of the set of regular economies $\mathcal{R}$. Equilibrium is unique for all endowment vectors ω in that component.*

Proof. Follows from the same property for the exchange model. See [2] or Proposition 8.8 in [7]. ☐

Let $\mathcal{R}_1$ denote the connected component of the set of regular economies $\mathcal{R}$ that contains the set P. A sufficient condition for the endowment vector ω to belong to $\mathcal{R}_1$ is that the distance of ω to the set P is small enough. A proxy for that distance is the length of the vector of net trades in factor contents.

Trade Theory: The two-country case

Proposition 12 can be improved into a complete characterization of economies with a unique equilibrium in the case of two countries.

Proposition 13. *For $m = 2$, the set of regular economies with a unique equilibrium is the component $\mathcal{R}_1$.*

Proof. By Theorem 5, it suffices to prove this property for the factor exchange model. That model having $m = 2$ consumers, the property follows from [3]. See [4] or [6] for two alternative proofs of the same property. ☐

It follows from Proposition 13 that if sufficiently small volumes of net trades in factor contents imply the uniqueness of equilibrium, large volumes generally imply the multiplicity of equilibria.

7. Demand functions resulting from utility maximization

So far, no use has been made of utility functions. Only demand functions have been considered. The goods demand function h_i has been shown to induce a factor demand function f_i. This was enough to define the factor exchange model associated with the Heckscher-Ohlin model and for proving the equivalence of these two models. This approach has undeniably the advantage of simplicity. Nevertheless, some readers may feel frustrated by the absence of any reference to preferences or even utility functions. In this section, I try to fill in this void by assuming that consumer i's preferences are represented by a utility function $u_i : X \to \mathbb{R}$ that satisfies the following properties: 1) Smoothness; 2) Smooth monotonicity, i.e., $Du_i(x_i) \in X$ for $x_i \in X$ where $Du_i(x_i)$ is the gradient vector defined by the first-order derivatives of u_i; 3) Smooth strict quasi-concavity, namely, the restriction of the quadratic form defined by the Hessian matrix $D^2u_i(x_i)$ to the tangent hyperplane to the indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ through x_i is negative definite; 4) The indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ is closed in $\mathbb{R}^k$ for all $x_i \in X$. These properties are standard in the literature on smooth economies. See, for example, [7], Chapter 2.

Maximizing the utility $u_i(x_i)$ with $x_i \in X$ subject to the budget constraint $q^T x_i \leq w_i$ then has the unique solution $h_i(q, w_i)$. It is well-known that the demand function $h_i : X \times \mathbb{R}_{++} \to X$ satisfies (S), (W), (H), (A), (WARP) and (ND). (See [7], Section 3.4.)

The following Proposition shows us that the induced demand function for goods f_i then results from the budget constrained maximization of some utility function for factors v_i that is induced by the utility function u_i for goods.

Proposition 14. *Consumer i's utility for factors induced by the utility for goods $u_i : \mathbb{R}^k_{++} \to \mathbb{R}$ is the function $v_i : \mathbb{R}^\ell_{++} \to \mathbb{R}$ defined by*

$$v_i(y_i) = \min_{p \in S} u_i\big(h_i(B(p)^T p, p^T y_i)\big).$$

for $y_i \in \mathbb{R}^\ell_{++}$.

Proof. Step 1. With $\hat{u}_i(q, w_i)$ denoting consumer i's indirect utility function for goods, Roy's identities with respect to $\hat{u}_i$ and the demand function h_i takes the form:

$$\partial_{w_i} \hat{u}_i(q, w_i) h_i(q, w_i) = -\partial_q \hat{u}_i(q, w_i).$$

Left multiplication by $B(p)$ yields

$$\partial_{w_i} \hat{u}_i(q, w_i) B(p)\, h_i(q, w_i) = -B(p)\, \partial_q \hat{u}_i(q, w_i),$$

which, after substituting $f_i(p, w_i) = B(p) h_i(q, w_i)$, yields

$$\partial_{w_i} \hat{u}_i(q, w_i) f_i(p, w_i) = -B(p)\, \partial_q \hat{u}_i(q, w_i) \tag{15}$$

Step 2. Define the function $\hat{v}_i : S \times \mathbb{R}_{++} \to \mathbb{R}$ by

$$\hat{v}_i(p, w_i) = u_i\big(h_i(B(p)^T p, w_i)\big) = \hat{u}_i(q, w_i),$$

with $q = B(p)^T p$. The derivative with respect to w_i of $\hat{v}_i(p, w_i)$ is equal to

$$\partial_{w_i} \hat{v}_i(p, w_i) = \partial_{w_i} \hat{u}_i(B(p)^T p, w_i) = \partial_{w_i} \hat{u}_i(q, w_i). \tag{16}$$

Similarly, the derivative with respect to p yields after application of the chain rule

$$\partial_p \hat{v}_i(p, w_i)^T = \partial_q \hat{u}_i(q, w_i)^T \; D(B(p)^T p).$$

Equality $D(B(p)^T p) = B(p)^T$ of Proposition A.7 (iii) of the Appendix implies

$$\partial_p \hat{v}_i(p, w_i)^T = \partial_q \hat{u}_i(q, w_i)^T \; B(p)^T$$

and, after transposition,

$$\partial_p \hat{v}_i(p, w_i) = B(p)\, \partial_q \hat{u}_i(q, w_i)\, . \tag{17}$$

Step 3.

Substitution of (16) and (17) in (15) yields the equality

$$\partial_{w_i} \hat{v}_i(p, w_i)\, f_i(p, w_i) = -\partial_p \hat{v}_i(p, w_i),$$

which is precisely Roy's identity for the indirect utility function $\hat{v}_i$ and the induced demand function f_i.
Step 4. One concludes by observing that the direct utility function v_i associated with the indirect utility function $\hat{v}_i$ is such that

$$v_i(y_i) = \min_{p \in S} \hat{v}_i(p, p^T y_i). \qquad \square$$

A brief application to the $2 \times 2 \times 2$ case

If countries' (consumers) utility functions for goods satisfy the standard assumptions recalled at the beginning of this Section, the associated demand functions for goods h_i satisfy (S), (W), (H), (A), (WARP) and (ND). From Section 5, the induced demand functions for factors f_i satisfy the same properties. This implies that the properties of the Heckscher-Ohlin model stated in Section 6 are also satisfied under the restriction that total factor resources are constant, the reason being that these properties are then satisfied by the associated factor exchange model. (It is an open problem whether this property is satisfied for demand functions that do not result from the maximization of transitive preferences, i.e., preferences representable by utility functions.)

Assume that total resources in factors are fixed and that countries' (consumers') preferences are defined by utility functions. Lancaster [18] and Dixit and Norman [13], pp. 108–110 then associate a 2×2 Edgeworth box diagram in the factor space with a $2 \times 2 \times 2$ Heckscher-Ohlin model. Nevertheless, that

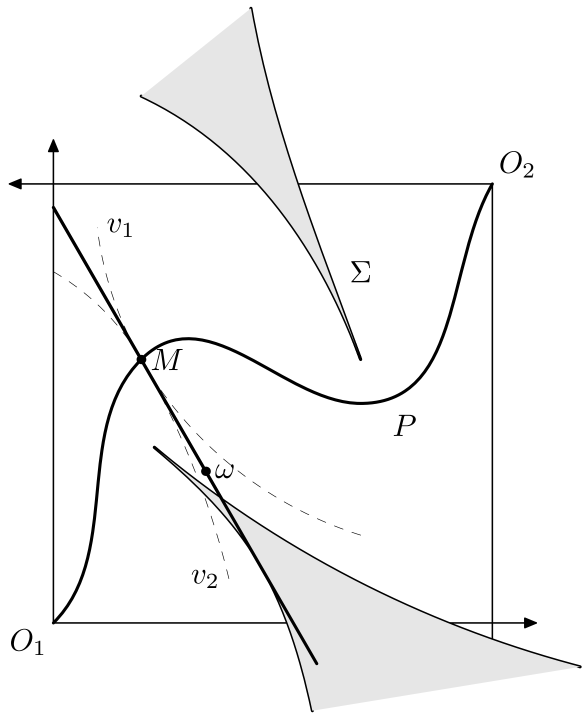

Figure 2: Edgeworth box diagram with indifference curves for factors

diagram differs from the usual Edgeworth box diagram for exchange economies because it contains no indifference curves. The definition of consumers' induced utility functions for factors and their properties (or, more precisely, the properties of the associated factor demand functions) considerably enrich the Lancaster-Dixit-Norman picture. In Figure 2, the point ω represents the endowments in factors of the two consumers. Equilibrium is unique at the point ω of the Figure and the point M represents the factor contents of the (unique) equilibrium allocation associated with ω. The point M is located on the contract curve P. The vector ωM then represents the net trade in factor contents. Note that the number of equilibria is equal to three for endowment vectors $\omega \in \mathcal{R}$ belonging to the shaded areas of Figure 2.

8. Concluding comments

This paper underlines at the theory level the essential role played in international trade by the volume of net trades in factor contents. Large volumes are associated with multiple equilibria and singular economies. Discontinuities of equilibrium selection maps at those singular economies translates into a high degree of volatility of world factor prices.

In a totally different direction, the New and New New Trade Theories have enriched the Heckscher-Ohlin model with countries having access to different technologies and with production that is subject to increasing returns to scale. These additional layers of complexity prevent the corresponding models from being equivalent to simpler exchange models. Nevertheless, numerous examples of complex models whose study has been illuminated by the properties of less complex versions suggest that the study of those models of international trade is likely to benefit from the understanding of the Heckscher-Ohlin model gained through its equivalence with an exchange model.

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A. Basic definitions and properties

A.1. Consumers' demand functions for goods

The following properties (S), (W), (H), (A), (WARP) and (ND) are defined only for demand functions for goods h_i. Similar definitions apply also to demand functions for factors f_i.

Definition A.1. *Let $h_i : X \times \mathbb{R}_{++} \to X$ be a demand functions for goods in the Heckscher-Ohlin model.*

i) (S) Smoothness: h_i is smooth;

ii) (W) Walras law: $q^T h_i(q, w_i) = w_i$ for any $(q, w_i) \in X \times \mathbb{R}_{++}$;

iii) (H) Homogeneity of degree zero: $h_i(\lambda q, \lambda w_i) = h_i(q, w_i)$ for every $\lambda > 0$ and any $(q, w_i) \in X \times \mathbb{R}_{++}$;

iv) (A) Boundary behavior: Let $(q^t, w_i^t) \in X \times \mathbb{R}_{++}$ be a sequence converging to $(q^0, w_i^0) \in \mathbb{R}^k_+ \times \mathbb{R}_{++}$ with $q^0 \neq 0$ but with some coordinates equal to 0, then $\limsup \|h_i(q^t, w_i^t)\| = +\infty$.

v) (WARP) weak axiom of revealed preferences:

$$\left.\begin{array}{r}(q')^T h_i(q, w_i) \leq w_i' \\ h_i(q, w_i) \neq h_i(q', w_i')\end{array}\right\} \implies q^T h_i(q', w_i') > w_i;$$

vi) (ND) Negative definiteness of the truncated Slutsky matrix: The Slutsky matrix truncated to its first $k-1$ rows and columns is negative definite at any $(q, w_i) \in X \times \mathbb{R}_{++}$.

A.2. Production

Production functions

The production function $F^j : \mathbb{R}^{\ell}_{+} \to \mathbb{R}_{+}$ is continuous; takes the value 0 on the boundary. In addition, F^j is smooth, monotone (i.e., $\partial F^j/\partial \eta^{\kappa} > 0$ for $1 \leq \kappa \leq \ell$), homogeneous of degree one and concave (constant returns to scale), with Hessian matrix $D^2F^j(\eta)$ negative semi-definite and of rank $\ell - 1$ on $\mathbb{R}^{\ell}_{++}$.

The following properties of production functions are well-known. They are here for reader's convenience.

Proposition A.2.

i) $DF^j(\eta)^T\eta = F^j(\eta) \neq 0$ *for* $\eta \in \mathbb{R}^{\ell}_{++}$;
ii) $\eta^T D^2F^j(\eta) = D^2F^j(\eta)\,\eta = 0$;
iii) The kernel of matrix $D^2F^j(\eta)$ *is the one-dimensional subspace generated by* $\eta \in \mathbb{R}^{\ell}_{++}$;
iv) The strict inequality $z^T D^2F^j(\eta)\, z < 0$ *is satisfied for any non-zero vector* $z \in \mathbb{R}^{\ell}$ *that is not collinear with* η;
v) The bordered Hessian matrix $\begin{bmatrix} D^2F^j(\eta) & DF^j(\eta) \\ DF^j(\eta)^T & 0 \end{bmatrix}$ *is invertible.*

Proof. i) $DF^j(\eta)^T\eta = F^j(\eta) \neq 0$ *for* $\eta \in \mathbb{R}^{\ell}_{++}$. By homogeneity of degree one, it comes $F^j(\lambda\eta) = \lambda F^j(\eta)$ with $\lambda \in \mathbb{R}$. It then suffices to take the derivative with respect to λ (Euler's identity). The inequality $DF^j(\eta)^T\eta \neq 0$ for $\eta \in \mathbb{R}^{\ell}_{++}$ follows from $F^j(\eta) \neq 0$.
ii) $\eta^T D^2F^j(\eta) = D^2F^j(\eta)\eta = 0$. The first order partial derivatives of F^j are homogeneous of degree zero. It then suffices to apply Euler's identity to these derivatives.
iii) Kernel of matrix $D^2F^j(\eta)$ *collinear with* $\eta \in \mathbb{R}^{\ell}_{++}$. The rank of $D^2F_j(\eta)$ is equal to $\ell - 1$. Its kernel is therefore one dimensional. One concludes by observing that the kernel also contains the vector $\eta = (\eta^1, \dots, \eta^{\ell}) \neq 0$.
iv) The strict inequality $z^T D^2F^j(\eta)\, z < 0$ *is satisfied for any non-zero vector* $z \in \mathbb{R}^{\ell}$ *not collinear with* η. All the ℓ eigenvalues of the symmetric matrix $D^2F^j(\eta)$ are real. A set of ℓ two by two orthogonal eigenvectors corresponds to these eigenvalues. Furthermore, the vector η can be chosen as the eigenvector associated with the eigenvalue 0. The $\ell - 1$ remaining eigenvalues are then strictly negative because of the rank assumption. Their associated eigenvectors generate a hyperplane that is orthogonal to the vector η. The restriction to that hyperplane of the quadratic form defined by matrix $D^2F^j(\eta)$ is therefore negative definite.

The vector $z \in \mathbb{R}^{\ell}$ is the sum $z = z' + z''$ of its orthogonal projections z' and z'', with z' in the vector space generated by η and z'' in the hyperplane orthogonal to η. It comes $z^T D^2F^j(\eta)z = (z'')^T D^2F^j(\eta)z''$ since z' is collinear with the vector η. The strict inequality $(z'')^T D^2F^j(\eta)z'' < 0$ then follows from $z'' \neq 0$ whenever z is not collinear with η.
v) Bordered Hessian matrix $\begin{bmatrix} D^2F^j(\eta) & DF^j(\eta) \\ DF^j(\eta)^T & 0 \end{bmatrix}$ *invertible.* Assume the contrary. There exists a vector $z = (\bar{z}, z^{\ell+1}) \neq 0 \in \mathbb{R}^{\ell} \times \mathbb{R}$ such that

$$\begin{bmatrix} D^2F^j(\eta) & DF^j(\eta) \\ DF^j(\eta)^T & 0 \end{bmatrix} \begin{bmatrix} \bar{z} \\ z^{\ell+1} \end{bmatrix} = 0.$$

This equality can be rewritten as

$$D^2F^j(\eta)\bar{z} + z^{\ell+1}DF^j(\eta) = 0 \tag{18}$$
$$\bar{z}^T DF^j(\eta) = 0. \tag{19}$$

Left multiplication of (18) by $\bar{z}^T$ yields, given (19),

$$\bar{z}^T D^2F^j(\eta)\,\bar{z} = 0.$$

By (iii), the vector $\bar{z}$ is therefore collinear with η: $\bar{z} = \lambda\eta$ with $\lambda \in \mathbb{R}$. By (ii), $D^2F^j(\eta)\,\bar{z} = 0$, so that (18) becomes $z^{\ell+1}DF^j(\eta) = 0$. It then follows from $DF^j(\eta) \neq 0$, itself a consequence of (i), that $z^{\ell+1} = 0$. This implies $z = \lambda(\eta, 0) \in \mathbb{R}^{\ell+1}$. Equation (19) becomes $\lambda\eta^T DF^j(\eta) = 0$, which is equivalent to $\lambda DF^j(\eta)^T\eta = 0$. Combined with (i), this implies $\lambda = 0$, a contradiction with $z \neq 0$. □

Isoquants

The set $\{\eta \in \mathbb{R}^\ell_{++} \mid F^j(\eta) = 1\}$ is the analog for ℓ factors of the textbook isoquants with two factors.

Proposition A.3.

i) The set $\{\eta \in \mathbb{R}^\ell_{++} \mid F^j(\eta) \geq 1\}$ is strictly convex;
ii) Its recession cone is the non-negative orthant $\mathbb{R}^\ell_{+}$.

Proof. i) Strict convexity of $\{\eta \in \mathbb{R}^\ell_{++} \mid F^j(\eta) \geq 1\}$. Let η and η' in $\mathbb{R}^\ell_{++}$ be such that $F^j(\eta) = F^j(\eta') = 1$. The vector η and η' are not collinear. Otherwise, assume $\eta' = \lambda\eta$ with $\lambda \neq 1$. Then, we would have $1 = F^j(\eta') = F^j(\lambda\eta) = \lambda F^j(\eta) = \lambda$, a contradiction with $\lambda \neq 1$.

The second derivative of the function $t \in [0,1] \to F^j((1-t)\eta + t\eta')$ is equal to $(\eta' - \eta)^T D^2F^j((1-t)\eta + t\eta')(\eta' - \eta)$ and is strictly negative by Proposition A.2, (iv) because $\eta' - \eta$ is not collinear with η. This implies the strict concavity of the function $t \in [0,1] \to F^j((1-t)\eta + t\eta')$, hence the strict inequality $F^j((1-t)\eta + t\eta') > 1$ for $t \in (0,1)$ and, therefore, the strict convexity of the set $\{\eta \in \mathbb{R}^\ell_{++} \mid F^j(\eta) \geq 1\}$.

ii) Recession cone. The vector $d \in \mathbb{R}^\ell$ defines a direction of recession for the set $\{\eta \in \mathbb{R}^\ell_{++} \mid F_j(\eta) \geq 1\}$ if, for some η^* in that set, the set $\{\eta^* + \alpha d \mid \alpha \geq 0\}$ is also contained in that set. This is equivalent to having $F_j(\eta^* + \alpha d) \geq 1$ for $\alpha \geq 0$. This is obviously satisfied by the monotonicity and continuity of F_j for $d \in \mathbb{R}^\ell_{+}$, which proves that the recession cone contains the non-negative orthant.

Conversely, let $d \in \mathbb{R}^\ell$ with at least one strictly negative coordinate. There is no loss of generality in assuming $d^1 < 0$. Let $\alpha > 0$ be defined by $\eta^{*1} + \alpha d^1 = 0$. Let $\alpha^t > \alpha$ be a sequence with $\lim_{t\to\infty}\alpha^t = \alpha$. Then, $\lim_{t\to\infty} F^j(\eta^* + \alpha^t d) = 0$, which contradicts the inequality $F^j(\eta^* + \alpha^t d) \geq 1$ and d cannot be a direction of recession. □

Cost functions and factor demand functions of the production sector

The following properties of the demand function for factors b_j and cost function σ_j in order to produce one unit of good j are also well-known, at least for the case of two factors. In this part, the factor price vector $p \in \mathbb{R}^\ell_{++}$ is not normalized.

Proposition A.4.

i) There is a unique factor bundle $b_j(p) \in \mathbb{R}^\ell_{++}$ that minimizes the cost of producing one unit of good j given the factor price vector $p \in \mathbb{R}^\ell_{++}$;
ii) The function $b_j : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_{++}$ is smooth and homogeneous of degree zero;
iii) The cost function $\sigma_j : \mathbb{R}^\ell_{++} \to \mathbb{R}$ is smooth and homogeneous of degree one;
iv) The cost function σ_j is concave.

Proof. i) Existence and uniqueness of the solution to the constrained cost minimization problem. Let $\eta^* \in \mathbb{R}^\ell_{++}$ be such that $F^j(\eta^*) \geq 1$. Adding the constraint $p^T\eta \geq p^T\eta^*$ has no effect on the solutions of the constrained cost minimization problem. The set defined by the two constraints is not only closed as the intersection of two closed sets, it is also bounded for obvious reasons. It follows from the compactness of that set and the continuity of the cost function that a solution exists to the constrained cost minimization problem.

The constraint $F^j(\eta) \geq 1$ is obviously binding by the monotonicity of F^j. The proof that the solution is unique is standard and proceeds by contradiction. Let $\eta \neq \eta'$ be two different solutions. By definition, the equalities $p^T\eta = p^T\eta'$ and $F^j(\eta) = F^j(\eta') = 1$ are satisfied. Let $\eta'' = (\eta + \eta')/2$. It follows from the strict concavity of the production function F^j restricted to a line not going through the origin that the strict inequality $F^j(\eta'') > 1$. It then suffices $\eta''' = \lambda\eta''$ such that $F^j(\eta''') = 1$ to get a contradiction to the definition of η and η' as cost minimizers.
ii) Homogeneity and smoothness of the factor demand functions. Homogeneity of degree zero of the demand function for factor $p \to b_j(p)$ is obvious. Smoothness follows from the application of the implicit function theorem to the first order conditions. These conditions take the form

$$\begin{cases} DF^j(\eta) - \mu q = 0, \\ F^j(\eta) - 1 = 0. \end{cases}$$

It is well-known that they are necessary and sufficient given the concavity of F^j. Smoothness then follows from Proposition A.2, (v).
iii) The cost function σ_j is smooth and homogeneous of degree one. Homogeneity is obvious. Smoothness follows from the cost function being the product of two smooth functions.
iv) The cost function σ_j is concave. Let $p \neq p'$ in $\mathbb{R}^\ell_{++}$. For $t \in [0, 1]$, the two inequalities

$$\sigma_j(p) = p^T b_j(p) \leq p^T b_j\big((1-t)p + tp'\big)$$
$$\sigma_j(p') = (p')^T b_j(p') \leq (p')^T b_j\big((1-t)p + tp'\big)$$

follow from the definitions of $\sigma_j(p)$ and $\sigma_j(p')$ as cost minimizing for p and p' respectively. Multiplication by $(1-t)$ and t of the first and second inequalities respectively followed by adding them up yields

$$(1-t)\sigma_j(p) + t\sigma_j(p') \leq \sigma_j\big((1-t)p + tp'\big). \qquad \square$$

Remark 2. The above proof can easily be adapted to show that if p and p' are not collinear and $t \in (0, 1)$, then, inequality (A.2) is strict.

Proposition A.5.

i) $p^T b_j(p) < p^T b_j(p')$ *for* p *and* p' *not collinear;*
ii) $p^T \dfrac{\partial b_j}{\partial p_\kappa}(p) = 0$ *for* $1 \leq \kappa \leq \ell$*;*
iii) $b_j = D\sigma_j$*;*
iv) $(p - p')^T(b_j(p) - b_j(p')) \leq 0$*;*
v) The Jacobian matrix $Db_j(p)$ *defines a negative semidefinite quadratic form;*
vi) $Db_j(p)^T p = 0$*;*
vii) The Jacobian matrix $Db_j(p)$ *has rank* $\ell - 1$*;*
viii) Let $p^0 = \lim_{t\to\infty} p^t$ *where* $p^t \in \mathbb{R}^\ell_{++}$ *(non-normalized prices) and* $p^0 \neq 0$ *has some coordinates equal to* 0*. Then,* $\limsup_{t\to\infty} \|b_j(p^t)\| = +\infty$*.*

Proof.
i) Follows from the definition of $b_j(p)$ as cost minimizing.
ii) From (i), the function $p' \to p^T b_j(p'$ is minimal at $p' = p$. The first order derivatives at $p' = p$ are equal to 0.
iii) The derivative of $\sigma_j(p) = p^T b_j(p)$ with respect to p_κ is equal to $\dfrac{\partial \sigma_j(p)}{\partial p_\kappa} = p^T \dfrac{\partial b_j}{\partial p_\kappa}(p) + b_j^\kappa(p)$.
It then suffices to apply (ii).
iv) Follows from the combination of the inequalities $p^T b_j(p) \leq p^T b_j(p')$ and $(p')^T b_j(p') \leq (p')^T b_j(p)$.
v) The derivation of that property from *(iv)* is standard.

vi) $Db_j(p)^T p = 0$. Follows readily from Euler's identity applied to $b_j(p)$, a function that is homogeneous of degree zero.
vii) The idea of the proof is to show that any vector $v \neq 0 \in \mathbb{R}^\ell$ in the kernel of $Db_j(p)$ is collinear to the factor price vector $p \in \mathbb{R}^\ell_{++}$. From the first order conditions satisfied by $b_j(p)$, it comes $DF^j(b_j(p)) - \mu(p)p = 0$ where $\mu(p) \neq 0$.

Taking the derivative of this equality with respect to the price vector p yields

$$D^2F^j(b_j(p))Db_j(p) = \mu(p)I + p(D\mu)^T$$

with I the $\ell \times \ell$ identity matrix. Right multiplication of this equality by $v \neq 0$ in the kernel of $Db_j(p)$ yields

$$\mu(p)v = -p\big((D\mu)^T v\big)$$

where $(D\mu)^T v$ is a real number. This equality implies that the vector $v \neq 0$ is necessary collinear with the factor price vector p.
viii) Step 1. One sees readily that The set $\{y \in \mathbb{R}^\ell_{++} \mid F^j(y) = 1 \text{ and } y \leq (A, \dots, A)\}$ is bounded away from zero for $A > 0$.
Step 2. The proof now proceeds by contradiction. Let $y^t = b_j(p^t)$ and assume $\limsup_{t\to\infty} \|y^t\| < +\infty$. This is equivalent to the sequence $\|y^t\|$ being bounded. There exists a real number $A > 0$ such that the inequalities $0 \leq y^t \leq (A, A, \dots, A)$ are satisfied for all t. Recall that $F^j(y^t) = 1$. Therefore, there exists by Step 1 $y_A \in \mathbb{R}^\ell_{++}$ such that $y_A \leq y^t \leq A$ for all t. By considering if necessary a subsequence, there is no loss in generality by assuming that the sequence y^t converges to some y^0 that satisfies the inequalities $y_A \leq y^0 \leq A$. By continuity, it comes $F^j(y^0) = 1$. In addition, the price vector p^t is collinear with the gradient vector $DF^j(y^t)$, i.e., there exists $\lambda^t > 0$ such that $p^t = \lambda^t DF^j(y^t)$. The sequences p^t and $DF^j(y^t)$ are bounded from above and bounded away from zero: therefore, the sequence λ^t is also bounded from above and away from zero. Considering once more and if necessary a subsequence, the sequence λ^t converges to some $\lambda^0 > 0$. It then follows from the continuity of DF^j that, at the limit, $p^0 = \lambda^0 DF^j(y^0)$. The contradiction comes from some coordinates of p^0 being equal to zero while each partial derivative of the production function F^j is different from zero. □

The production matrix

Definition A.6. *The production matrix associated with the (non normalized) factor price vector* $p \in \mathbb{R}^\ell_{++}$ *is the* $\ell \times k$ *matrix* $B(p) = \big[b_1(p)\ b_2(p)\ \dots\ b_k(p)\big]$.

The following properties of the matrix function $p \to B(p)$ extend those of the demand functions for factors of the production sector.

Proposition A.7.

- *i)* *The function* $p \to B(p)$ *is smooth.*
- *ii)* $DB(p)^T p = 0$.
- *iii)* $D(B(p)^T p) = B(p)^T$.
- *iv)* $p^T B(p) < p^T B(p')$ *for* p *and* p' *not collinear.*

Proof. *(i)* Follows from Proposition A.4 (ii).
(ii) Follows readily from Proposition A.5 (vi).
iii) The derivative of the matrix product $B(p)^T p$ is equal to $D(B(p)^T p) = B(p)^T + DB(p)^T p$. One concludes by applying (i).
iv) Follows readily from Proposition A.5 (i). □

B. A lemma about embeddings

An embedding $\phi : X \to Y$ is a smooth map between two smooth manifolds X and Y that is an immersion (its derivative map $D\phi(x) : T_x X \to T_{f(x)} Y$ between the tangent spaces $T_x X$ and $T_{f(x)} Y$ is into, i.e., an injection) and also a homeomorphism between its domain X and its image $\phi(X)$. A very nice feature of embeddings is that the image $\phi(X)$ is then also a smooth submanifold of the range Y. Embeddings provide a very convenient way of proving that some subset $\phi(X)$ of the smooth manifold Y is actually a smooth submanifold of Y. The global structure of the smooth submanifold $\phi(X)$ as homeomorphic to X then comes as a courtesy. The application of the following lemma requires little more than the computation of derivatives (i.e., Jacobian matrices).

Lemma B.1. *Let $\phi : X \to Y$ and $\psi : Y \to X$ be two smooth mappings between smooth manifolds with:*

i) $\psi : Y \to X$ *is onto (i.e., a surjection);*
ii) $\phi \circ \psi = \mathrm{id}_Y$.

Then, $Z = \phi(X))$ is a smooth submanifold of Y diffeomorphic to X.

Proof. The strategy is to show that the smooth map $\phi : X \to Y$ is an embedding, which therefore implies that its image $Z = \phi(X)$ is a submanifold of Y diffeomorphic to X.

To prove the homeomorphism part, we first observe that ϕ, viewed as a map from X to $Z = \phi(X)$, is a surjection. To prove that ϕ is an injection, assume $\phi(x) = \phi(x')$. Since $\psi : Y \to X$ is onto, there exist y and y' with $x = \psi(y)$ and $x' = \psi(y')$. It comes $\phi(x) = \phi \circ \psi(y) = y$ and $\phi(x') = \phi \circ \psi(y') = y'$, hence $y = y'$. This proves that ϕ viewed as a map from X to Z is a continuous bijection.

Let $\psi | Z$ denote the restriction of the map ψ to the subset Z of Y. The relation $\psi \circ \phi = \mathrm{id}_Y$ implies $(\psi | Z) \circ \phi = \mathrm{id}_Y$ from which follows that the inverse map of ϕ (as a map between X and Z) is $\psi | Z$. The maps $\phi : X \to Y$ and $\psi : Y \to X$ are continuous (in fact, smooth). It follows readily from the definition of the induced topology of Z that the restriction $\psi | Z : Z \to X$ is also continuous as well as the map (also denoted by) $\phi : X \to Z = \phi(X)$. (Note that the fact that Z is simply a subset of Y equipped with the induced topology does not make it a "nice" subset of Y yet, which prevents us from using the above argument to infer that $\psi | Z : Z \to X$ and $\phi : X \to Z$ are smooth mappings.) At the moment, these two maps are just continuous. They therefore define inverse homeomorphism between X and Z.

To prove the immersion part, take $y \in Y$. Let $x = \psi(y)$. The relation $\phi \circ \psi = \mathrm{id}_Y$ yields, by taking its derivative,

$$D\phi(x) \circ D\psi(y) = \mathrm{id}_{T_y(Y)}$$

where $T_y(Y)$ denotes the tangent space to the manifold Y at y. This relation implies that the linear map between tangent spaces $D\psi(y) : T_y(Y) \to T_x(X)$ is an injection. The map $\phi : X \to Y$ is therefore an immersion. In combination with the homeomorphism part above, this proves that the map $\phi : X \to Y$ is an embedding. $\square$

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